Assignment 7

Due Friday 10 April at the beginning of class

This Assignment is based on the slides, especially weeks 10 & 11 and week 12.

DO THE FOLLOWING PROBLEMS.

P17. *First see the "main idea" about the definition of $C_c^\infty(\Omega)$ in the weeks 10 & 11 slides.*

(a) Suppose $\Omega \subset \mathbb{R}^d$ is open. Suppose $g \in L^1_{loc}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Prove that $gD^\alpha\varphi \in L^1(\Omega)$, thus that $\int_\Omega gD^\alpha\varphi\,dm$ is well-defined.

(b) Show that $C(\Omega) \subset L^1_{loc}(\Omega)$. Then, for each dimension d, construct an example of Ω and $f \in C(\Omega)$ showing that "$C(\Omega) \subset L^1(\Omega)$" is false.

P18. *Here is an important case using the weak derivative concept in 1D.*

Suppose $\Omega = (a, b)$ is a bounded open interval, and let $a = t_0 < t_1 < \cdots < t_n = b$ be a mesh on a closed interval $[a, b]$. As in the slides, the P_1 finite element space is defined as the set of functions which are continuous on $[a, b]$, and which are piecewise linear on each cell (subinterval) $[t_{k-1}, t_k]$. Show that $P_1 \subset H^1(\Omega)$.

P19. *This problem is removed.*

P20. *This problem has revisions.*

Recall that the norm in $H^k(-\pi, \pi)$ satisfies $(\|f\|_{H^k})^2 = \sum_{\ell=0}^{k} \int_{-\pi}^{\pi} |f^{(\ell)}(x)|^2\,dx$. Here $f^{(\ell)}$ denotes the weak derivative of order ℓ.

(a) Suppose $f \in H^1(-\pi, \pi)$ satisfies $f(-\pi) = f(\pi)$. Using the formulas for Fourier series,[1] prove the derivative formula

$$f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} inc_n e^{inx},$$

with convergence in L^2, where c_n are the Fourier coefficients of f.

(b) We say $f \in H^k(S^1)$, where S^1 is the unit circle, if $f \in H^k(-\pi, \pi)$ and if $f^{(\ell)}(-\pi) = f^{(\ell)}(\pi)$ for all $0 \le \ell < k$. Show that if $f \in H^k(S^1)$ then

$$\|f\|_{H^k}^2 = \sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} n^{2j} \right) |c_n|^2.$$

[1] See "recall Fourier series in $L^2(-\pi, \pi)$" in the week 12 slides.

P21. *This example asks you to do the details of an example shown in the slides. It shows that H^1 contains discontinuous functions for planar domains. To say it more precisely, there are equivalence classes in H^1 which have no continuous representative.*

Let $D = \{x \in \mathbb{R}^2 : |x|^2 < 1/4\}$ be an open disk, centered at the origin, with radius $1/2$. Let $g(x) = \ln(r)$ and $p(x) = \ln(|\ln(r)|)$, in polar coordinates where $|x| = r$, defined a.e. on D. (*These functions are not defined at the origin.*) Exactly compute, by hand, the L^2 and H^1 norms of g and p. Then use a computer to plot g and p over $0 < r < 1/2$.

P22. For $z = x + iy$ a complex number, define $p(z) = z^3 - 5z^2 + 2z - 1$. Show that the real and imaginary parts of p, regarded as real-valued functions of (x, y) on $\mathbb{R}^2$, are harmonic.