Assignment 6

Due Friday 27 March at the beginning of class

This Assignment is based on Chapters 3 and 5 of our textbook,[1] and on the slides, especially those for weeks 10 & 11.

DO THE FOLLOWING Exercises from Chapter 5 (see pages 129–134):

- **Exercise 5.1.1**
- **Exercise 5.2.1**

DO THE FOLLOWING ADDITIONAL PROBLEMS.

P12. *Use the classical definition of absolute continuity, and the FTC2, from the slides.*

Suppose throughout that $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous.

(a) Show that fg is also absolutely continuous.

(b) Show that the product rule $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$ holds almost everywhere.

(c) Show the following integration-by-parts formula, justifying the existence of all (Lebesgue) integrals and the existence of point values:

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

P13. *Use the product rule for partial derivatives of scalar functions, as stated in the slides.*

(a) Assume that $\Omega \subset \mathbb{R}^d$ is open. Suppose that $f : \Omega \to \mathbb{R}$ and $\mathbf{V} : \Omega \to \mathbb{R}^d$ are differentiable at every point of Ω. Using the definition of the divergence and gradient, show this product rule:

$$\nabla \cdot (f\,\mathbf{V}) = \nabla f \cdot \mathbf{V} + f\,\nabla \cdot \mathbf{V}.$$

(b) Assume that $\Omega = (0,1)^d \subset \mathbb{R}^d$ is the open unit cube. Suppose that $f, g \in C_c^1(\Omega)$. Show the "integration-by-parts 1: partial derivatives" rule in the slides, namely

$$\int_\Omega \frac{\partial f}{\partial x_k} g\,dm = -\int_\Omega f \frac{\partial g}{\partial x_k}\,dm,$$

with careful attention to why there is no integral over $\partial\Omega$ in this rule.

P14. *Basic properties of the Fourier transform make good exercises with the $L^1(\mathbb{R}^1)$ space. There is a more advanced analysis, the Plancherel theorem, which shows that in fact the transform is a Hilbert space isometric isomorphism of $L^2(\mathbb{R}^1)$, but that would take us too far afield.*

[1] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

Suppose $f : \mathbb{R}^1 \to \mathbb{C}$ is measurable. The *Fourier transform* of f is

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dx, \tag{1}$$

for those $t \in \mathbb{R}^1$ such that the integral exists, in which case $\hat{f}(t) \in \mathbb{C}$.

(a) Suppose $f \in L^1(\mathbb{R}^1)$. Show that $\hat{f}(t)$ is well-defined for every t, and that $\|\hat{f}\|_\infty \leq \frac{1}{\sqrt{2\pi}}\|f\|_1$. (*Thus* $\hat{f} \in L^\infty(\mathbb{R}^1)$.)

(b) Again suppose $f \in L^1(\mathbb{R}^1)$. Use Lebesgue's dominated convergence theorem (DCT) to show that $\hat{f}$ is uniformly continuous. (*Thus* $\hat{f} \in C(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1)$.)

(c) Suppose $f \in L^1(\mathbb{R}^1)$, and also that $g(x) = -ixf(x)$ is in $L^1(\mathbb{R}^1)$. Use the DCT to show that $\hat{f}$ is differentiable at every t, and that $\hat{f}'(t) = \hat{g}(t)$. (*Thus* $\hat{f} \in C^1(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1)$.)

P15. Suppose $f \in L^1([0,1])$ and define the *Cesàro operator*

$$(Tf)(s) = \frac{1}{s}\int_0^s f(t)\, dt.$$

(a) Show that T is linear, and determine a kernel $k(s,t)$ so that T can be written as a Fredholm integral operator, in the form $(Tf)(s) = \int_0^1 k(s,t)\, f(t)\, dt$.

(b) Show that $(Tf)(s)$ is continuous at $s > 0$. (*Hint.* DCT.)

(c) Give an example of $f \in L^1([0,1])$ which shows that Tf need not be continuous at $s = 0$. The same example should show that $Tf \notin L^\infty([0,1])$.

P16. Suppose $\rho = \rho(t,x) = \rho(t, x_1, x_2, x_3)$, for $t \in \mathbb{R}^1$ and $x \in \mathbb{R}^3$, is continuously-differentiable (in each variable). Suppose $\mathbf{J} = \mathbf{J}(t,x) = \mathbf{J}(t, x_1, x_2, x_3)$ is also continuously-differentiable. We may interpret ρ as a scalar mass density, or charge density etc., and $\mathbf{J}$ as a vector-valued mass flux. One says that mass is conserved if the PDE

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{2}$$

applies. Now suppose that $\Omega \subset \mathbb{R}^3$ is any open and bounded set such that $\partial\Omega$ is continuously-differentiable. Apply the divergence theorem to (2) to show

$$\frac{d}{dt}\left(\int_\Omega \rho\, dm\right) = -\int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n}\, ds. \tag{3}$$

Interpret (3) physically. (*What is the total mass inside* Ω*? How does that quantity change? Relate to the net normal flux across the boundary. Note that* $\mathbf{n}$ *is the* outward *normal direction.*)