

Assignment 5

Due Wednesday 18 March at the beginning of class

This Assignment is based on Chapter 3 of our textbook,¹ especially sections 3.3, 3.4, and 3.6, and on the lectures. However, please also read sections 5.1 and 5.2; this will be the material which we cover next.

DO THE FOLLOWING Exercises from Chapter 3 (see pages 69–73):

- **Exercise 3.3.4**
- **Exercise 3.4.1**
- **Exercise 3.4.2**
- **Exercise 3.6.4** *This problem assumes that p and q are Hölder conjugate exponents: $\frac{1}{p} + \frac{1}{q} = 1$. (Hint. Rewrite this simple definition in some different ways.) Also, I wrote my proof in terms of a single function $\phi = f + g$. If you do this, justify it.*
- **Exercise 3.6.7**

DO THE FOLLOWING ADDITIONAL PROBLEMS.

P7. *The following problem asks for the heart of the correct proof of Theorem 3.22, in the $p = 1$ and \mathbb{R}^1 case. The book's definition of step functions is wrong, but the Theorem is true.*

A function $h : X \rightarrow \mathbb{R}$ is a *step function* if it is a finite linear combination of characteristic functions of intervals; there are coefficients $c_k \in \mathbb{R}$ and intervals $I_k \subset \mathbb{R}$ so that

$$h(x) = \sum_{k=1}^n c_k \mathbb{1}_{I_k}(x).$$

Suppose $E \subset [0, 1]$ is measurable. Show that the indicator function $f(x) = \mathbb{1}_E(x)$ can be arbitrarily approximated in L^1 norm by a step function. That is, show that for all $\epsilon > 0$ there exists a step function $g(x)$ so that

$$\|\mathbb{1}_E - g\|_1 = \int_{[0,1]} |\mathbb{1}_E(x) - g(x)| dm < \epsilon.$$

(Hint. Return to the definition of $\mathcal{M}_{\mathcal{F}}$. Use it. The function g is also an indicator.)

P8. *(Simplified.)* Prove that the triple (X, \mathcal{R}, μ) given in Example 6 at the beginning of Section 3.6 (page 58) is, in fact, a measure space.

P9. *(Simplified: part (b) removed.)* Suppose that (X, \mathcal{R}, μ) is a measure space. We say that two measurable functions $f, g : X \rightarrow \mathbb{R}$ are *equal almost everywhere* if the set of points where they differ has μ -measure zero. Prove that this is an equivalence relation on the set of such measurable functions.

¹K. Saxe, *Beginning Functional Analysis*, Springer 2010.

P10. Suppose that (X, \mathcal{R}, μ) is a *finite* measure space, that is, suppose that $X \in \mathcal{R}$ and $\mu(X) < \infty$. For $f : X \rightarrow \mathbb{C}$ measurable (with respect to \mathcal{R}) and $1 \leq p < \infty$, define $L^p(X, \mu)$ and $\|f\|_p$ as usual; see page 59. Consider two different powers, $1 \leq p < q < \infty$. Show that $L^q(X, \mu) \subset L^p(X, \mu)$. Give an example of a function on $(X, \mu) = ([0, 1], m)$, which will depend on p and q , **which shows that** this is a proper inclusion.

P11. *The requested proofs will help you appreciate Hölder's and Minkowski's inequalities. These results are used in Theorems 3.17, 3.18, and 3.19. For part (a), start with the triangle inequality on \mathbb{R} , and also $|u| + |v| \leq 2 \max(|u|, |v|)$. A hint on part (b) is on page 60.*

(a) Suppose $1 \leq p < \infty$ and let $u, v \in \mathbb{R}$. Show that

$$|u + v|^p \leq 2^p (|u|^p + |v|^p).$$

(b) *(Fixed.)* Suppose $1 < p, q < \infty$ are (Hölder) conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$. Also suppose $x, y \geq 0$. Prove *Young's inequality*:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Extra Credit. Suppose $1 \leq p < q < \infty$, and consider ℓ^p and ℓ^q . Prove that for $f \in \ell^1$,

$$\|f\|_\infty \leq \|f\|_q \leq \|f\|_p \leq \|f\|_1.$$

Also prove that $\ell^p \subset \ell^q$ —this inclusion reverses the direction from **P10**—and give an example to show that this is a proper inclusion.