Assignment 3

Due Monday 16 February at the beginning of class

This Assignment is based on Chapters 2 and 4 of our textbook,[1] and on the lectures. When you do calculations from Chapter 4 you will use the L^2 space. However, knowledge of the Lebesgue integral will not be necessary; you may regard integrals as Riemann integrals if you wish.

DO THE FOLLOWING Exercises from Chapter 2 (see pages 29–31):

- **Exercise 2.3.4**
- **Exercise 2.3.7** *Take ℓ^1 to be real sequences. First argue that if a sequence $(a_n) \subset \ell^1$ is Cauchy then for fixed location k the entries are also Cauchy—$(a_n^k)_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$—so we get the entries of a limiting sequence a. The next question is whether $a \in \ell^1$? To show this, bound the relevant partial sums by a constant. Then take a limit, and show that this limit has the same bound. Complete the proof by showing that $\|a_n - a\|_1 \to 0$.*

DO THE FOLLOWING Exercises from Chapter 4 (see pages 88–91):

- **Exercise 4.1.1**
- **Exercise 4.1.3** *For part* **(b)***, note that "converges in mean" is synonymous with "converges in $\|\cdot\|_2$", and please justify the use of a theorem.*
- **Exercise 4.1.4** *For all parts, generally your examples $\{f_n\}$ do* not *need to be continuous, but of course $f_n \in L^2$ is required. Part* **(c)** *needs the fact that $[-\pi, \pi]$ is of finite length.*
- **Exercise 4.2.2** *You may assume that "L^2" refers to $L^2([a,b])$ for $a, b \in \mathbb{R}$, but the statement holds generally, for any $L^2(X, \mu)$.*
- **Exercise 4.2.3**
- **Exercise 4.2.5** *This calculation was mostly done in the week 4 slides.*

DO THE FOLLOWING ADDITIONAL PROBLEMS.

P4. *Chapter 2 defines* equicontinuous *and proves the Ascoli-Arzelà Theorem. However, it gives no related exercises! Here is one.*

(a) Consider $V = C([0,1])$ with the supremum norm $\|\cdot\|_\infty$. For $\mu, \alpha \in (0,1)$ let

$$f_{\alpha,\mu}(x) = \begin{cases} 0, & 0 \le x \le \alpha \\ (x-\alpha)/\mu, & \alpha \le x \le \alpha + \mu \\ 1, & \alpha + \mu \le x \le 1 \end{cases}$$

[1] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

Sketch the graphs of $f_{\alpha,\mu}$ in three cases of your choice. (*Generally* μ *should be small.*) Explain why $f_{\alpha,\mu} \in V$. Let

$$E = \{f_{\alpha,\mu} \,:\, \mu, \alpha \in (0,1)\} \subset V.$$

Show that E is *not* equicontinuous.

(b) Now define

$$g_{\alpha,\mu}(x) = \int_0^x f_{\alpha,\mu}(t)\,dt,$$

and let $H = \{g_{\alpha,\mu} \,:\, \mu, \alpha \in (0,1)\}$. Sketch the graphs of $g_{\alpha,\mu}$ in the same three cases as in part **(a)**. Then show that H *is* an equicontinuous subset of V.