Assignment 1

Due Friday 23 January ← *revised*

This Assignment is based on lectures and on Chapter 1 of our textbook.[1]

DO THE FOLLOWING Exercises for Chapter 1 (pages 13–15):

- **Exercise 1.1.1**
- **Exercise 1.1.3**
- **Exercise 1.1.5**
- **Exercise 1.1.10**
- **Exercise 1.3.2**
- **Exercise 1.3.3**

DO THE FOLLOWING ADDITIONAL PROBLEMS.

P1. Consider the real vector space $C([0,1])$ and the linear functional

$$\ell[f] = \int_0^1 f(x)\, e^x \, dx,$$

that is, $\ell : C([0,1]) \to \mathbb{R}$.

(a) Show ℓ is linear.

For a sequence of functions $\{f_k(x)\}_{k=1}^\infty \subset C([0,1])$ we define the real sequence $\ell_k = \ell[f_k] \in \mathbb{R}$.

(b) For $k \in \mathbb{N}$ compute ℓ_k if $f_k(x) = x^k$.

(c) For $k \in \mathbb{N}$ compute ℓ_k if $f_k(x) = \sin(k\pi x)$.

Comment (revised). For a finite linear combination, $g(x) = \sum_{k=1}^n c_k f_k(x)$, we have $\ell[g] = \sum_{k=1}^n c_k \ell_k$ by linearity, and from part **(b)** this defines ℓ on all polynomials. If $\{f_k(x) = x^k\}$ were a "basis" then the values $\{\ell_k = \ell[f_k]\}$ would "represent" ℓ as an infinite-length vector. However, correctly defining "basis" for infinite-dimensional spaces is rather difficult! The best kind of "basis" is the *complete orthonormal sequence* (Chapter 4), an but we are not there yet; also see the footnote on page 11. Problem **P1** only requires that you figure out how to do some integrals, and you don't have to prove anything is a basis. We will eventually see that if we change the vector space to a Hilbert space then this ℓ *does* have a representation, and it is represented either by the function e^x itself or the coefficients on a complete orthonormal sequence.

P2. Show that $a_k = 1/k$, $k \in \mathbb{N}$, is in ℓ^2, and find its norm. Is it in ℓ^1? Is it in ℓ^∞?

[1] Saxe (2010) *Beginning Functional Analysis*, Springer.