Assignment 5

Due Monday 1 April 2024 *(revised)*

This Assignment is based primarily on sections 3.1, 3.2, and 3.3 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

P24. *Here is why one always assumes that $\mathcal{D}(T) = \mathcal{H}$ for a <u>bounded</u> operator.*

After the text defines "operator" (Definition 3.1; page 36) it says that "a bounded operator admits a unique continuous extension to the full space $\mathcal{H}$, since $\mathcal{D}(T)$ is dense." Prove this.

P25. *This is a basic exercise for Definition 3.4, of the operator adjoint.*

(a) Consider the unbounded multiplication operator M_a on $\mathcal{H} = \ell^2 = \ell^2(\mathbb{N})$, defined for $x = (x_1, x_2, x_3, \dots)$ as

$$M_a x = (x_1, 2x_2, 3x_3, \dots),$$

with a domain consisting of sequences which are eventually zero:

$$\mathcal{D}(M_a) = \left\{x \in \ell^2 \,:\, \text{there is } N \text{ so that if } k \geq N \text{ then } x_k = 0\right\}.$$

Show that $\mathcal{D}(M_a)$ is a dense subspace of $\mathcal{H}$.

(b) Directly from Definition 3.4, find a formula for the adjoint M_a^*, and its domain.

(c) Let M_b be the unbounded multiplication operator with the same formula $M_b x = (x_1, 2x_2, 3x_3, \dots)$ but on the larger domain

$$\mathcal{D}(M_b) = \left\{x \in \ell^2 \,:\, \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty\right\}.$$

Show that $\mathcal{D}(M_b)$ is dense and that M_b is self-adjoint (Definition 3.19).

P26. *Please read Example 3.6 on page 39. In this exercise you prove a crucial aspect.*

(a) Let $\mathcal{V} = C^1[0,1]$ be a normed vector space with norm $\|v\| = \left(\int_0^1 |v(x)|^2\,dx\right)^{1/2}$. Define $\omega : \mathcal{V} \to \mathbb{C}$ by $\omega(v) = v(0)$. Show that ω is linear, but also show that it is *not* bounded.

In the next part you do <u>not</u> need to prove the assertion that $C^1[0,1]$ is dense in $L^2[0,1]$, <u>nor</u> do you need to prove that ℓ is linear.

(b) Let $\mathcal{H} = L^2[0,1]$ and $\mathcal{D}(T) = C^1[0,1]$ for $T = d/dx$ the first derivative. Since $\mathcal{D}(T)$ is dense in $\mathcal{H}$, thus T is an operator by Definition 3.1. Fix $u \in \mathcal{D}(T)$ and define the linear functional $\ell : \mathcal{D}(T) \to \mathbb{C}$ by

$$\ell(v) = -\langle Tu, v\rangle + \overline{u(1)}v(1) - \overline{u(0)}v(0).$$

Show that if ℓ is bounded in the $\mathcal{H}$ norm then $u(0) = u(1) = 0$.

P27. *This is a simplification of Exercise 3.12. For both parts below note that f, being merely measurable, could be unbounded, and indeed it could go to infinity anywhere in $(0,1)$, but at least $f(x) \in \mathbb{C}$ is well-defined for every $x \in (0,1)$. Also, note that we may conclude from part* **(b)** *that M_f is self-adjoint if and only if f is real a.e.*

Suppose $f : (0,1) \to \mathbb{C}$ is measurable. Let M_f be the multiplication operator on $\mathcal{H} = L^2(0,1)$ with domain

$$\mathcal{D}(M_f) = \left\{ v \in L^2(0,1) \,:\, fv \in L^2(0,1) \right\}$$

and formula

$$(M_f g)(x) = f(x)g(x)$$

for $g \in \mathcal{H}$.

(a) Show that $\mathcal{D}(M_f)$ is dense in $\mathcal{H}$.

(b) Show that $\mathcal{D}(M_f) = \mathcal{D}(M_f^*)$ and that $M_f^* = M_{\overline{f}}$.