# Assignment 4

## Due Wednesday 6 March 2024

This Assignment is based primarily on sections 2.6, 2.7, and 3.1 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

**P19.** *This is Exercise 2.10, but clarified. One way to show* **(a)** *is to argue that* $\mathrm{range}(P) = \ker(Q)$ *for some bounded operator* $Q \in \mathcal{L}(\mathcal{H})$ *built from* $P$.

Suppose $P \in \mathcal{L}(\mathcal{H})$ satisfies $P^2 = P$ and $\langle Pv, w\rangle = \langle v, Pw\rangle$ for all $v, w \in \mathcal{H}$.

**(a)** Show that $\mathrm{range}(P)$ is a closed subspace of $\mathcal{H}$.

**(b)** Show that $\mathcal{H} = \mathrm{range}(P) \oplus \ker(P)$.

**P20.** *This is Exercise 2.6, asking for the proof of Corollary 2.29. Note that "bounded sesquilinear form" is defined on page 27. Observe that both* $\langle\cdot,\cdot\rangle$ *and* $\eta(\cdot,\cdot)$ *are conjugate linear in the first spot. Prove the linearity and boundedness of* $T$. *Hint: Start by showing that for fixed* $w \in \mathcal{H}$, *the functional* $\ell(v) = \overline{\eta(v, w)}$ *is in* $\mathcal{H}'$.

Given a bounded sesquilinear form $\eta : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, show that there is a unique operator $T \in \mathcal{L}(\mathcal{H})$ so that

$$\eta(v, w) = \langle v, Tw\rangle$$

for all $v, w \in \mathcal{H}$.

**P21.** Carefully prove Corollary 2.36, Parseval's theorem and equality, from Theorems 2.34 and 2.35.

**P22.** *In Example 2.32 on page 28 our textbook casually says that a "standard argument using the Dirichlet kernel" shows that a certain set of orthogonal complex exponentials is in fact a basis. This exercise begins this substantial argument, which was perhaps the greatest triumph of 19th century analysis. Specifically, we define the Dirichlet kernel and look at its properties. Note that* $\mathcal{H} = L^2(-\pi, \pi)$ *here, instead of* $L^2(0, 2\pi)$ *as in the book, but this is a mere convenience.*

**(a)** Let $\phi_k(\theta) = \dfrac{1}{\sqrt{2\pi}} e^{ik\theta}$ for $k \in \mathbb{Z}$. Observe that these functions are continuous on $[-\pi, \pi]$, thus in $\mathcal{H} = L^2(-\pi, \pi)$. Show that $\{\phi_k\}$ is an orthonormal set.

**(b)** For $n \in \mathbb{N}$, let $D_n(\theta) = \sum_{j=-n}^{n} e^{ij\theta}$ be the Dirichlet kernel. Show that $D_n(0) = 2n+1$ (*easy!*). For $\theta \neq 0$ show that

$$D_n(\theta) = \frac{\sin\left((n+1/2)\theta\right)}{\sin(\theta/2)}.$$

(*Hint. Use a geometric series argument, and the fact that* $\sin\alpha = (e^{i\alpha} - e^{-i\alpha})/(2i)$.)

**(c)** Use a computer to plot $D_2(\theta)$, $D_8(\theta)$, and $D_{20}(\theta)$ in a single figure, on the interval $-\pi \le \theta \le \pi$.

**(d)** Compute $\int_{-\pi}^{\pi} D_n(\theta)\,d\theta$. Conjecture what the integral $\int_{-\pi}^{\pi} D_n(\theta) f(\theta)\,d\theta$ yields when $f(x)$ is continuous at $x = 0$ and $n$ is large.

**(e)** Show that if $f \in \mathcal{H} = L^2(-\pi,\pi)$ then

$$\int_{-\pi}^{\pi} D_n(x-\theta) f(\theta)\,d\theta = 2\pi \sum_{j=-n}^{n} \langle \phi_j, f\rangle\, \phi_j(x).$$

Argue that therefore

$$(P_n f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-\theta) f(\theta)\,d\theta$$

is the orthogonal projection of $f \in \mathcal{H}$ onto $\operatorname{span}\{\phi_{-n}, \dots, \phi_n\}$. (*Hint. Compare (2.30).*)

**P23.** *Hints. Use Cauchy-Schwarz for part* **(a)**. *An example for part* **(b)** *can be built from a power of the polar coordinate* $r = (x^2+y^2)^{1/2}$.

**(a)** Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Suppose $K \in L^2(\Omega)$, that is,

$$\int_0^1 \int_0^1 |K(x,y)|^2\,dx\,dy < \infty.$$

Show that the integral operator with kernel $K$, namely

$$(T_K f)(x) = \int_0^1 K(x,y) f(y)\,dy,$$

is linear and bounded on $\mathcal{H} = L^2(0,1)$. That is, $T_K \in \mathcal{L}(\mathcal{H})$.

**(b)** Find $K \in L^2(\Omega)$ which is *not* a bounded function.

**(c)** Fix a finite sequence $a_j \in \mathbb{C}$, $j = 1, \dots, n$. Let

$$K_a(x,y) = \sum_{j=1}^{n} a_j e^{i\,2\pi j(x-y)}.$$

This is a continuous and bounded function and thus $K_a \in L^2(\Omega)$. Find all of the eigenvalues and eigenfunctions of the operator $T_{K_a} \in \mathcal{L}(\mathcal{H})$.