Assignment 2

Due Monday 12 February 2024 (*revised!*), at the start of class

This Assignment is based primarily sections 2.3, 2.4, 2.6, and 2.7 of our textbook,[1] but see also sections 2.1 and 2.2, and the handout.

DO THE FOLLOWING EXERCISES.

P8. *This is Exercise 2.1. Note that "bounded" is defined on page 9 and "continuous" was defined on the handout.*

For normed vector spaces $\mathcal{V}$ and $\mathcal{W}$, prove that a linear map $T : \mathcal{V} \to \mathcal{W}$ is bounded if and only if it is continuous.

P9. *This is Exercise 2.5. Note that $\|T\|$ is defined on page 9.*

For $T \in \mathcal{L}(\mathcal{H})$, prove that

$$\|T\| = \sup_{v,w\neq 0} \frac{|\langle v, Tw\rangle|}{\|v\|\,\|w\|}.$$

P10. *This is Exercise 2.7. Weak convergence of a sequence in $\mathcal{H}$ is defined on page 27. You may use Corollary 2.36.*

Let $\mathcal{H}$ be a Hilbert space and suppose $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set. Prove that the sequence (e_n) converges weakly to 0.

P11. Prove directly, without using the Heine-Borel theorem, that the set

$$K = \{0\} \cup \left\{\tfrac{1}{n} \,:\, n \in \mathbb{N}\right\} \subset \mathbb{R}$$

is compact in the usual topology on $\mathbb{R}$.

[1] D. Borthwick (2020). *Spectral Theory: Basic Concepts and Applications*, Springer

P12. *This example is so important that I did it in class* and *I want you to write out the details! You may use, without comment, the standard properties of integration, as they apply to functions in $L^1(0,1)$.*

(a) Consider the Banach space $\mathcal{V} = L^1(0,1)$ and the linear operator

$$(Af)(x) = \int_0^x f(t)\,dt$$

for $f \in \mathcal{V}$. Show that $Af \in \mathcal{V}$. Also show A is bounded.

By part **(a)**, *we may write $A \in \mathcal{L}(\mathcal{V})$.*

(b) Show that, in fact, if $f \in \mathcal{V}$ then Af is a continuous function on $[0,1]$. (*Show this directly, even though it is also stated in the handout as a fact. You may use result (A.6) in Appendix A, which is nearly stating what you are trying to prove.*) Observe that $(Af)(0) = 0$.

In the next part, you may use, without comment, the form of the Fundamental Theorem of Calculus in the handout. You may also use the fact that the only continuous functions $y(x)$ satisfying $y'(x) = \alpha y(x)$, for $\alpha \in \mathbb{C}$, on x in any non-trivial interval of the real line, are the functions $y(x) = ce^{\alpha x}$ for $c \in \mathbb{C}$.

(c) By definition, $f \in \mathcal{V}$ is an *eigenfunction* of A if $f \neq 0$ and there is $\lambda \in \mathbb{C}$ so that $Af = \lambda f$.[2] If f is an eigenfunction of A then we call the corresponding λ the *eigenvalue*. Show that A has no eigenvalues.

[2]Pay attention here. "$f \neq 0$" means f is not the zero vector of $\mathcal{V}$. Which means what about the pointwise values $f(x)$? Also, "$Af = \lambda f$" means what? (Think about *almost everywhere*.)