Assignment 1

Due Friday 26 January 2024, at the start of class (*revised & corrected*)

This Assignment is based primarily on the "Definitions and facts" handout, but also on sections 2.1 and 2.2 of our textbook

> D. Borthwick (2020). *Spectral Theory: Basic Concepts and Applications*, Springer

DO THE FOLLOWING EXERCISES.

P1. Suppose $(V, \|\cdot\|)$ is a normed vector space. Let $Y \subset V$ be any finite set of points. Show that Y is closed.

P2. Let $(V, \|\cdot\|)$ be a normed vector space. Let $S_1 = \{x \in V : \|x\| = 1\}$. Recalling that $B_1 = B_1(0)$ is the open ball of radius one centered at the origin, show S_1 is exactly the boundary set of B_1: $\partial B_1 = S_1$.

P3. Let X_k be the set of RNA sequences of length k. All you need to know about such things is that an RNA sequence has one character from $\{A, C, G, U\}$ in each location. That is, an element of X_k is a string of k letters, each of which is A, C, G, U. Let $\delta(a, b) = 0$ if $a = b$ and $\delta(a, b) = 1$ if $a \neq b$. For $x, y \in X_k$ let

$$d(x, y) = \sum_{j=1}^{k} \delta(x_j, y_j)$$

where x_j is the jth letter in the RNA sequence x.

(a) Show rigorously that (X_k, d) is a metric space. That is, show $d(x, y)$ computes a valid distance between any two RNA sequences of the same length.

(b) Can (X_k, d) be regarded as a subset of a real normed vector space $(V, \|\cdot\|)$? Argue, at least informally, that this is true. You will need to construct V and sketch the embedding. Formally, one says that there is an *isometric* embedding $X_k \hookrightarrow V$. (*Hint.* I used $\|\cdot\|_\infty$ on each letter.)

P4. Show that a convergent sequence in a metric space cannot converge to two different limits.

P5. **(a)** Is $f(x) = \dfrac{1}{\sqrt{x}}$ in $L^1(0, 1)$? In $L^2(0, 1)$?

(b) Let $s \geq 0$ and define $g_s(x) = \dfrac{1}{x^s}$. For $1 \leq p \leq \infty$, determine exactly which spaces $L^p(0, 1)$ contain g_s.

P6. *The definition of the spaces $\ell^p = \ell^p(\mathbb{N})$ in section 2.2 of the textbook is too terse. This problem at least includes a clear definition for $p = \infty$. Make sure to observe that infinite sequences are the same as functions from $\mathbb{N}$ to $\mathbb{C}$.*

The normed vector space $(\ell^\infty, \|\cdot\|_\infty)$ is the set of infinite sequences $a = (a_k) = (a_1, a_2, a_3, \dots)$ with complex entries ($a_k \in \mathbb{C}$) which are bounded, that is, so that there is $M > 0$ so that $|a_k| < M$ for all $k \in \mathbb{N}$. The norm is the supremum of the absolute values of the entries:

$$\|a\|_\infty = \sup_{k=1,2,\dots} |a_k|.$$

(a) Show that $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space.

(b) Consider the set of complex-valued sequences which have a limit of zero:

$$Y = \{(a_k) \,:\, a_k \to 0\}.$$

Show that $Y \subset \ell^\infty$.

(c) Show that $(Y, \|\cdot\|_\infty)$ is a Banach space. (*Hint.* You may do this directly or use part **(a)**. For less than obvious reasons, this subset Y is often instead called c_0.)

P7. For any set X and measure μ, the normed vector space $L^1(X, d\mu)$ is defined in section 2.2 of our textbook. For such a space the Minkowski (triangle) inequality (2.5) can be proved directly, without reference to Appendix A.2. Do so.