Definitions and facts leading to the spectral theorem

Page numbers are for Borthwick, *Spectral Theory* Springer 2020.

Notation: $\forall$="for all", $\exists$="there exists", $\mathcal{H}$ is a separable Hilbert space, T is an (unbounded) operator on $\mathcal{H}$, $T - z = T - zI$, $U \in \mathcal{L}(\mathcal{H})$ is a unitary operator, and A is an (unbounded) self-adjoint operator on $\mathcal{H}$.

def p 36 an *operator* T is a linear map on $\mathcal{H}$ with a dense domain $\mathcal{D}(T)$

def p 38 the *adjoint* of T is an operator T^*, with domain

$$\mathcal{D}(T^*) = \{v \in \mathcal{H} \,:\, \ell(u) = \langle v, Tu\rangle \in \mathcal{L}(\mathcal{H}, \mathbb{C})\},$$

so that $\langle T^*v, u\rangle = \langle v, Tu\rangle$ for all $v \in \mathcal{D}(T^*)$, $u \in \mathcal{D}(T)$

def p 41 an operator is *closed* if its graph is a closed subset of $\mathcal{H} \times \mathcal{H}$

fact p 43 the adjoint T^* is always closed

fact p 44 $T = T^{**}$ if T is closed

fact p 44 T closable $\iff$ $\mathcal{D}(T^*)$ dense

fact p 44 **closed graph theorem.** when $\mathcal{D}(T) = \mathcal{H}$: T closed $\iff$ $T \in \mathcal{L}(\mathcal{H})$

def p 46 T has *bounded inverse*: $\exists\, T^{-1} \in \mathcal{L}(\mathcal{H})$ s.t. $TT^{-1} = I$ on $\mathcal{H}$ and $T^{-1}T = I$ on $\mathcal{D}(T)$

fact p 46 $T^{-1} \in \mathcal{L}(\mathcal{H}) \iff T$ is closed, T is bounded away from zero, and $\operatorname{range}(T)$ dense

def p 47 A is *self-adjoint* if $A^* = A$ $\leftarrow$ **requires:** $\mathcal{D}(A^*) = \mathcal{D}(A)$

def p 47 T is *symmetric* if $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $v \in \mathcal{D}(T)$

fact p 47 T is symmetric $\implies$ T is closable

fact p 47 A is self-adjoint $\implies$ A is symmetric

def p 47 T is *positive* if $\langle v, Tv\rangle \geq 0$ for all $v \in \mathcal{D}(T)$

def p 67 *eigenvalue* and *eigenvector*: $T\phi = \lambda\phi$ for $\phi \in \mathcal{D}(T) \setminus \{0\}$ and $\lambda \in \mathbb{C}$

def p 68 *spectrum*: the set $\sigma(T) = \{\lambda \in \mathbb{C} \,:\, T - \lambda$ does not have a bounded inverse$\}$

def p 68 *resolvent set*: $\rho(T) = \mathbb{C} \setminus \sigma(T)$

def p 68 if $z \in \rho(T)$ then $R_z = (T - z)^{-1}$ is the *resolvent* operator

fact p 68 if T is not closed then $\sigma(T) = \mathbb{C}$

fact p 69 if T is bounded then $\sigma(T) \subset B_{\|T\|}(0)$

fact p 69 $\sigma(T^*) = \sigma(T)^*$, $\rho(T^*) = \rho(T)^*$, and $\left[(T - z)^{-1}\right]^* = (T - \overline{z})^{-1}$

fact p 71 for $f : X \to \mathbb{C}$ measurable and M_f a multiplication operator on $L^2(X, d\mu)$:
$\lambda \in \mathbb{C}$ is an eigenvalue of $M_f \iff \mu(f^{-1}(\lambda)) > 0$

def p 71 $\text{ess-range}\, f = \left\{z \in \mathbb{C} \,:\, \mu(f^{-1}(B_\epsilon(z))) > 0\, \forall \epsilon > 0\right\}$

fact p 71 $\sigma(M_f) = \text{ess-range}\, f$

fact p 71 $\|(M_f - z)^{-1}\| = \Big(\operatorname{dist}\left(z, \sigma(M_f)\right)\Big)^{-1}$

fact p 83 if T closed then $\rho(T)$ is open and $R_z = (T - z)^{-1}$ is analytic in z on $\rho(T)$

fact p 85 if T is bounded then $\sigma(T) \neq \emptyset$

def p 85 *spectral radius*: $r(T) = \sup_{z \in \sigma(T)} |z|$

fact p 85 if T bounded then $r(T) \le \|T\|$

fact p 86 $\sigma(A) \subset \mathbb{R}$ ← **recall notation:** A is self-adjoint

fact p 87 $z \in \sigma(A) \iff \exists \{u_n\} \subset \mathcal{D}(A)$ s.t. $\|u_n\| = 1$ and $\|(A-z)u_n\| \to 0$

def p 17 U is *unitary* if it is bijective and an isometry (i.e. $\|Ux\| = \|x\| \, \forall x \in \mathcal{H}$)

fact p 17 U unitary $\iff$ U bijective & $\langle Ux, Uy \rangle = \langle x, y \rangle \; \forall x, y \in \mathcal{H}$

fact p 102 U unitary $\iff$ $U \in \mathcal{L}(\mathcal{H})$ and $UU^* = U^*U = I$

def p 102 *functional calculus*: on T we can apply a function $f : \mathbb{C} \to \mathbb{C}$ to create an operator $f(T)$

def p 102 $\mathbb{S} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ and $C(\mathbb{S}) = \{f : \mathbb{S} \to \mathbb{C} \,|\, f$ is continuous (and periodic)$\}$

fact p 103 **continuous functional calculus for unitaries.** fix U unitary. there is a map $C(\mathbb{S}) \to \mathcal{L}(\mathcal{H})$, $f \mapsto f(U)$ so that

(0) if $f(z) = 1$ then $f(U) = I$
(a) $f(U)^* = \overline{f}(U)$
(b) $f(U)g(U) = (fg)(U)$ ← **thus:** $f(U)g(U) = g(U)f(U)$
(c) if $f \ge 0$ then $f(U) \ge 0$
(d) $\|f(U)\| = \sup_{z \in \mathbb{S}} |f(z)|$

def p 105 if X is a metric space then $C(X) = \{f : X \to \mathbb{C}$ continuous$\}$

def p 105 $\beta : C(X) \to \mathbb{C}$ is *positive* if $f \ge 0 \implies \beta(f) \ge 0$

fact p 105 **Riesz representation theorem.** suppose X is a compact metric space and $\beta : C(X) \to \mathbb{C}$ is linear and positive. there is a unique positive Borel measure on X so that

$$\beta(f) = \int_X f \, d\mu \qquad \forall f \in C(X)$$

def p 105 for U unitary and $v \in \mathcal{H}$ the *spectral measure* is μ_v on $\mathbb{S}$ so that $\langle v, f(U)v \rangle = \int_{\mathbb{S}} f \, d\mu_v$

fact p 106 for μ from the Riesz representation theorem, $C(X) \subset L^2(X, \mu)$ is dense

fact p 107 **spectral theorem for unitaries.** if $\mathcal{H}$ is a separable Hilbert space and $U \in \mathcal{L}(\mathcal{H})$ is unitary then there is a countable collection of finite measures ν_k on $\mathbb{S}$, and a measurable space $(Y, \nu) = \cup_k (\mathbb{S}, \nu_k)$, and a unitary map $W : L^2(Y, \nu) \to \mathcal{H}$, so that

$$W^{-1} U W = M_\eta$$

where $M_\eta \in \mathcal{L}(L^2(Y, \nu))$ is a bounded multiplication operator and $\eta : Y \to \mathbb{C}$ is equal to $\eta(z) = z$ on each copy of $\mathbb{S}$

def p 108 the *Cayley transform* $\gamma(z) = \dfrac{z - i}{z + i}$ maps $\mathbb{R}$ to $\mathbb{S}$

fact p 108 **spectral theorem (multiplication operator form).** if $\mathcal{H}$ is a separable Hilbert space and A is self-adjoint on $\mathcal{H}$ then there is a countable collection of finite Borel measures μ_k on $\mathbb{R}$, and a measurable space $(X, \mu) = \cup_k (\mathbb{R}, \mu_k)$, and a unitary map $Q : L^2(X, \mu) \to \mathcal{H}$, so that

$$Q^{-1} A Q = M_\alpha$$

where $M_\alpha \in \mathcal{L}(L^2(X, \mu))$ is a (generally unbounded) multiplication operator and $\alpha : X \to \mathbb{R}$ is equal to $\alpha(x) = x$ on each copy of $\mathbb{R}$