Review Guide

for in-class **Midterm Quiz 2** on **Monday 13 April**

The second Midterm will cover new material since the first Midterm. I am calling this assessment a "Quiz" because it will be shorter than Midterm Exam 1. It will cover some new material from the textbook,[1] plus a number of topics from the "calculation" slides.[2] Specifically, Midterm Quiz 2 will cover:

- new material from textbook sections 3.4, 3.6, 5.1, and 5.2,
- lecture material on dual spaces, the Hilbert projection and decomposition theorems, and the Riesz representation theorem,
- the weeks 10 & 11 slides, and
- the week 12 slides.

However, you will **not** need to know non-trivial ideas about partial differential equations (PDEs) like the Poisson equation.

Inevitably you will need to know some material covered on the first Midterm. This includes the basics of normed vector spaces, inner product spaces, completeness, the Lebesgue integral, and the spaces $C([a,b])$ and ℓ^p. There is some overlap between the material on this Review Guide and the the previous Guide. However, **I do not intend to ask questions which specifically cover material before week 8 and Midterm Exam 1**.

The problems will be of the following types; see the lists below: state definitions, state theorems, describe or illustrate geometrical ideas, apply theorems in easy situations, and prove simple theorems/corollaries.

Definitions. Be able to state the precise definition. Be able to prove ideas which follow immediately from the definitions.

- abstract measure space $(X, \mathcal{R}, \mu)$
- almost everywhere
- $\int_A f\, d\mu$, the (abstract) Lebesgue integral of a real-valued or complex-valued function f over A with respect to the measure μ
- for $1 \le p < \infty$, the vector space $L^p(X,\mu)$ is the set of equivalence classes of functions, up to almost everywhere, such that $\int_X |f|^p\, d\mu < \infty$, with norm $\|f\|_p = \left(\int_X |f|^p\, d\mu\right)^{1/p}$
- essentially-bounded measurable function
- the vector space $L^\infty(X,\mu)$ is the set of equivalence classes of essentially-bounded functions, up to almost everywhere, with the essential bound as the norm $\|f\|_\infty$

[1] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

[2] The slides are linked on the Daily Log tab of the public course page.

- complete orthonormal (ON) sequence in an inner product space
- Fourier coefficients and Fourier series with respect to an ON sequence
- absolute continuity (classical definition for functions on $\mathbb{R}^1$), and $AC([a,b])$
- $\text{supp}(f)$, for a real-valued or complex-valued function on an open set $\Omega \subset \mathbb{R}^d$
- $C_c(\Omega)$, $C_c^k(\Omega)$, $C_c^\infty(\Omega)$, on open sets $\Omega \subset \mathbb{R}^d$
- $D^\alpha f$, multi-index notation for partial derivatives on $\mathbb{R}^d$
- $L^1_{loc}(\Omega)$, on open sets $\Omega \subset \mathbb{R}^d$
- weak derivatives of order α, for functions on open sets $\Omega \subset \mathbb{R}^d$
- bounded linear operator, between normed vector spaces
- $\mathcal{B}(X, Y)$, for normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$
- operator norm, making $\mathcal{B}(X, Y)$ into a normed vector space
- dual space X', for a normed vector space $(X, \|\cdot\|_X)$
- Laplacian $\nabla^2 u$, on differentiable functions u on $\Omega \subset \mathbb{R}^d$
- harmonic function
- $H^k(\Omega)$, for $k = 0, 1, 2, \dots$ and an open set $\Omega \subset \mathbb{R}^d$
- $H^k(S^1)$ (there are two equivalent definitions; know both)
- $H^1_0(\Omega)$, for an open set $\Omega \subset \mathbb{R}^d$

Theorems and Lemmas. Know the precise statement of the theorem. Be able to prove if so indicated.

- Lebesgue's monotone convergence theorem (Theorem 3.13)
- Fatou's lemma (Theorem 3.14)
- Lebesgue's dominated convergence theorem (Theorem 3.15)
- for $1 \le p \le \infty$, $L^p(X, \mu)$ is a complete normed linear space (Theorems 3.17, 3.19, 3.21, and text on page 61 for L^∞)
- Hölder's inequality (Theorem 3.18)
- normed vector space is complete $\iff$ (absolute conv. $\implies$ conv.) (Theorem 3.20)
- step functions are dense in $L^p(X, \mu)$ (see Corrections tab of the public website)
- Fourier coefficients are $c_k = \langle f, f_k \rangle$ if $\{f_k\}$ is a complete ON sequence (Theorem 4.1)
- bounded $\iff$ continuous $\iff$ cont. at 0 (Theorems 5.1 & 5.2) **[be able to prove]**
- operator norm is a norm on $\mathcal{B}(X, Y)$ (Theorem 5.3) **[be able to prove]**
- $\mathcal{B}(X, Y)$ is a Banach space if Y is a Banach space (Theorem 5.4)
- corollary: dual spaces are Banach spaces **[be able to prove]**
- Hilbert projection theorem: if $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $C \subset X$ is a closed and convex subset then for every $x \in X$ there exists a unique $y \in C$ that minimizes $f(w) = \|x - w\|$ over $w \in C$
- Hilbert orthogonal decomposition theorem: if $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $M \subset X$ is a closed subspace then $X = M \oplus M^\perp$
- Riesz-Fréchet representation theorem: if $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $\ell \in X'$ then there exists a unique $x_\ell \in X$ so that $\ell(x) = \langle x, x_\ell \rangle$ for all $x \in X$

- fundamental theorem of calculus: if $f \in L^1(a,b)$ then $F(x) = \int_a^x f(t)\,dx$ is in $AC([a,b])$ and $F'(x) = f(x)$ a.e., and if $f \in AC([a,b])$ then $f' \in L^1(a,b)$ and $f(x) = f(a) + \int_a^x f'(t)\,dt$ (week 10&11 slides)
- divergence theorem: if $\Omega \subset \mathbb{R}^d$ is bounded and $\partial\Omega$ is C^1, and if $\mathbf{V} \in C^1(\overline{\Omega})$, then $\int_\Omega \nabla \cdot \mathbf{V}\,dm = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\,ds$ (week 10&11 slides)
- integration-by-parts on $(a,b) \subset \mathbb{R}^1$ (week 10&11 slides) **[be able to prove]**
- product rule 1 and 2 (week 10&11 slides)
- integration-by-parts 1 and 2 (week 10&11 slides) **[be able to prove]**
- Poincaré-Friedrichs inequality (week 12 slides)

Examples of linear functionals and linear operators. Be able to define these, *and determine whether they are bounded/continuous, or not, on various spaces.*

- integral against a fixed function $\left[\text{e.g. } \ell(f) = \int_\Omega fg\,dm \text{ where } g \in L^p(\Omega)\right]$
- point evaluation at a fixed point $\left[\text{e.g. } \ell(f) = f(x_0) \text{ where } f \in C(\Omega) \text{ and } x_0 \in \Omega\right]$
- left- and right-shift operator on ℓ^p
- integral operators, with given kernels

Corrections to the textbook. See the Corrections tab of the public website.