Review Guide

for in-class **Midterm Exam 1** on **Monday 2 March**

The first Midterm Exam will cover Chapters 1–4 in the textbook,[1] plus any closely-related ideas in the "calculation" slides.[2] Specifically, the Exam will cover sections 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 3.2, 3.3, 3.6, 4.1, 4.2, and 4.3. However, I will *not* ask questions from sections 3.1 (probability), 3.4 (Lebesgue convergence theorems), or 3.5 (Riemann-versus-Lebesgue); we have not yet covered these sections adequately. While I will not ask about any of the nontrivial proofs in section 3.6, please do *read the main ideas*.

The problems will be of the following types, based on the lists below: state definitions, state theorems, describe or illustrate geometrical ideas, apply theorems in easy situations, and prove simple theorems/corollaries.

Definitions. Be able to state the precise definition. Be able to prove ideas which follow immediately from the definitions.

- metric $d(\cdot,\cdot)$, and metric space
- discrete metric
- convergence of a sequence in a metric space
- continuity of a function $f:(M,d_M)\to(N,d_N)$, at a point $x_0\in M$
- norm $\|\cdot\|$, and normed vector space, both real and complex
- inner product $\langle\cdot,\cdot\rangle$, and inner product space, both real and complex[3]
- sequence space ℓ^2, and its inner product $\langle\cdot,\cdot\rangle$
- sequence spaces ℓ^p, for $1\le p<\infty$, including norm $\|\cdot\|_p$
- sequence space ℓ^∞, and its norm $\|\cdot\|_\infty$
- $C([a,b])$, continuous functions on $[a,b]$, real or complex, and the sup norm $\|\cdot\|_\infty$
- $C([a,b])$ as an inner product space with $\langle f,g\rangle=\int_a^b f(x)\overline{g(x)}\,dx$
- linear algebra which needs no topology: linear independent, span, ∞-dimensional
- balls in metric spaces
- limit points, isolated points, and interior points in metric spaces
- open and closed subsets of a metric space
- open covers, and subcovers, in metric spaces
- compact sets and sequentially compact sets in metric spaces

[1] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

[2] The slides are linked on the Daily Log tab of the public course page.

[3] In the textbook, complex inner products have the conjugate on the second variable, e.g. $\langle f,g\rangle=\int f(x)\overline{g(x)}\,dx$, and thus symmetry is $\langle u,v\rangle=\overline{\langle v,u\rangle}$, and linearity is in the first variable $\langle\lambda u,v\rangle=\lambda\langle u,v\rangle$.

- separable metric space
- Cauchy sequence (in a metric space)
- complete metric space
- Banach and Hilbert space
- ring of sets, and σ-ring of sets
- countably-additive functions on sets
- measure
- interval in $\mathbb{R}^n$
- $\mathcal{E}$ is the collection of all finite disjoint unions of intervals
- 2^X is the collection of all subsets of X, the power set
- outer measure m^* on arbitrary subsets of $\mathbb{R}^n$
- symmetric difference $S(A, B)$ of sets A, B
- $\mathcal{M}_\mathcal{F}$ is the collection of all subsets $A \subset \mathbb{R}^n$ for which there is a sequence $A_k \in \mathcal{E}$ so that $m^*(S(A_k, A)) \to 0$ as $k \to \infty$
- $\mathcal{M}$ is the collection of all subsets of $\mathbb{R}^n$ which can be written as finite or countable unions of sets in $\mathcal{M}_\mathcal{F}$
- measurable function $f : \mathbb{R}^n \to \mathbb{R}$
- (measurable) simple function
- Lebesgue integral of a (measurable) simple function
- Lebesgue integral of a (measurable) nonnegative function
- a real-valued function is integrable
- $\int_E f\, dm$, that is, the Lebesgue integral of f over E
- a set $A \subset \mathbb{R}^n$ has measure zero if $m^*(A) = 0$
- almost everywhere
- abstract measure space $(X, \mathcal{R}, \mu)$
- counting measure (on any set)
- for $1 \le p < \infty$, the vector space $L^p(X, \mu)$ is the set of equivalence classes of functions, up to almost everywhere, such that $\int_X |f|^p\, d\mu$ is finite
- for $1 \le p < \infty$, $\|f\|_p = \left(\int_X |f|^p\, d\mu\right)^{1/p}$ is a norm on $L^p(X, \mu)$
- integral of a complex-valued function
- orthogonal vectors in an inner product space
- orthonormal (ON) sequence
- Fourier coefficients and Fourier series with respect to an ON sequence
- complete orthonormal sequence

Theorems and Lemmas. Know the precise statement of the theorem. Be able to prove if so indicated.

- Cauchy-Schwarz (Theorem 1.3; **be able to prove** in the real case)
- arbitrary unions/intersections of open/closed sets (Theorem 2.2; **be able to prove**)

- compact sets in metric spaces are closed (Theorem 2.3; **be able to prove**)
- compactness $\iff$ sequentially compactness (Theorem 2.4; **be able to prove**)
- Heine-Borel theorem for $\mathbb{R}^n$: compact $\iff$ closed and bounded (Theorem 2.5)
- Arzelà-Ascoli theorem (Theorem 2.6)
- $C([a,b]), \|\cdot\|_\infty$ is separable (Section 2.2).
- $C([a,b]), \|\cdot\|_\infty$ is complete (Theorem 2.7)
- the sequence spaces ℓ^p, for $1 \le p < \infty$, are separable (**be able to prove**)
- the sequence spaces ℓ^p, for $1 \le p \le \infty$, are complete
- $\mathcal{M}$ is a σ-ring, and m^* is countably additive on $\mathcal{M}$ (Theorem 3.6)
- equivalent definitions of measurable (Theorem 3.7)
- algebra and limits preserve measurability (Theorem 3.9)
- approximating functions by simple functions (Theorem 3.10; **be able to prove**)
- basic properties of the Lebesgue integral (Theorems 3.11 and 3.12)
- for $1 \le p < \infty$, $L^p(X,\mu)$ is a normed linear space (Theorems 3.17 and 3.19)
- for $1 \le p < \infty$, $L^p(X,\mu)$ is complete (Theorems 3.21)
- Fourier coefficients are $c_k = \langle f, f_k \rangle$ (Theorem 4.1; **be able to prove**)
- Bessel's inequality (Theorem 4.2)
- the original Riesz-Fischer theorem (Theorem 4.4)
- the full-power Riesz-Fischer theorem from lecture, proven using the Gram-Schmidt process: separable Hilbert spaces H are isometrically isomorphic to ℓ^2
- equivalent conditions showing that an ON sequence is complete (Theorem 4.5)
- Fourier's sequence is complete; classical series converge in mean (Theorem 4.6)

Techniques and facts. Be able use these calculation/proof techniques, and justify if requested.

- Norms generate metrics (Theorem 1.1), inner products generate norms (Thm 1.2).
- To show infinite dimensional, find a linearly-independent set with n elements, for each $n \in \mathbb{N}$ (Section 1.3).
- The norm in a normed vector space, and the inner product in an inner product space, are continuous.

Corrections. There are 3 topics in the textbook[4] that need correction, to my knowledge.

1. On page 19, *equicontinuous at* $x \in [a,b]$, for a set E, is defined correctly. However, the given definition of "equicontinuous set on an interval" is not the correct hypothesis for Theorem 2.6. The textbook should say:

   > *Definition.* Let $E \subset C([a,b])$. The set E is *equicontinuous at* $x \in [a,b]$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $y \in [a,b]$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $f \in E$. The set E is *uniformly equicontinuous* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $f \in E$ and $x, y \in [a,b]$ and $|x-y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

[4] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

From this definition, Theorem 2.6 should say:

> *Theorem 2.6 (The Ascoli-Arzelà Theorem).* Let $E \subset \big(C([a,b]), \|\cdot\|_\infty\big)$. Then E is compact if and only if E is closed, bounded, and uniformly equicontinuous on $[a,b]$.

2. On page 46 of the textbook there is a minor omission, namely of a finite-measure assumption. (*I suppose that the author wants to avoid a complicated statement. However, without an addititional hypothesis the textbook's definition can generate* $\infty - \infty$ *ambiguities.*) The textbook should say:

   > *Definition.* Let $E \in \mathcal{M}$. For a measurable simple function $s(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{E_k}(x)$, such that $m(E \cap E_k) < \infty$ for each k, we define the *Lebesgue integral of s over E* by
   > $$\int_E s\,dm = \sum_{k=1}^{N} c_k m(E \cap E_k).$$
   > If $c_k \geq 0$ for all k then this integral gives a well-defined result in $[0, +\infty]$ even without the finite measure assumption.

3. The definition of a *step function* on page 67 is not correct, and the proof of Theorem 3.22 should be ignored. What the textbook says is nearly vacuous, whereas Riesz's result for Euclidean space, namely Theorem 3.22 when corrected, is significant.

   > *Definition.* Suppose $X \subset \mathbb{R}^n$ is an open subset or an interval. A function $f : X \to \mathbb{C}$, or $f : X \to \mathbb{R}$, is a *step function* if it is a finite linear combination of characteristic functions of intervals,
   > $$f(x) = \sum_{k=1}^{n} c_k \mathbb{1}_{I_k}(x),$$
   > with $c_k \in \mathbb{C}$ or $c_k \in \mathbb{R}$, respectively.

   Now the idea is that any L^p function can be approximated by a step function because step functions can approximate characteristic functions of measurable sets, that is, simple functions. This is true *if* the norm is an integral norm against Lebesgue measure m, which is why there is a $p < \infty$ restriction in the Theorem below.

   > *Theorem 3.22.* Suppose $X \subset \mathbb{R}^n$ is an open subset or an interval. If $1 \leq p < \infty$ then the step functions are dense in $L^p(X, m)$.

   Finally, note that the density of step functions is *not* automatic for general measure spaces $(X, \mathcal{R}, \mu)$; Theorem 3.22 is for Lebesgue measure.