Review Guide

for in-class **Midterm Quiz** on **Wednesday March 20**

The Midterm Quiz will be built from those parts of Chapter 2 in Borthwick[1] which we have actually covered, and from the Handout.[2] In Borthwick we covered 2.1, 2.2, 2.3, 2.4, 2.6, and 2.7 pretty thoroughly, except for 2.3.2. You will not be asked about 2.5 (Sobolev spaces) or 2.3.2.

The problems will be of these types, based on the lists below: state definitions, state theorems, describe or illustrate geometrical ideas, apply theorems in easy situations, and prove simple theorems/corollaries.

Definitions. Be able to state the precise definition.

- metric $d(\cdot,\cdot)$
- norm $\|\cdot\|$
- open and closed subsets of a metric space
- compact set
- limit of a sequence (in a metric space)
- continuous $\mathbb{C}$-valued function (on a metric space)
- Cauchy sequence (in a metric space)
- sequential compactness of a set (in a metric space; p. 30)
- complete metric space
- Banach space
- linear map between vector spaces
- eigenvector and eigenvalue for a linear map on a vector space
- inner product $\langle\cdot,\cdot\rangle$ on a complex vector space
- norm in an inner product space
- complex Hilbert space $\mathcal{H}$
- $\ell^p = \ell^p(\mathbb{N})$ spaces for $1 \le p \le \infty$, including norm
- positive measure
- almost everywhere
- simple function
- $L^p(X) = L^p(X, dm)$[3] for $1 \le p < \infty$, including norm
- $L^\infty(X) = L^\infty(X, dm)$, including norm

[1] D. Borthwick, *Spectral Theory: Basic Concepts and Applications*, GTM 284, Springer 2020.

[2] `bueler.github.io/fa/assets/handouts/getstarted.pdf`

[3] For $1 \le p \le \infty$ and $dm =$ (Lebesgue measure), $L^p(X) = L^p(X, dm)$ can be defined for any Lebesgue-measurable set $X \subset \mathbb{R}^n$. Note that $L^p(X, dm)$ is a set of *measurable* functions $f : X \to \mathbb{C}$.

- $C^\infty(\mathbb{R})$, $C^\infty(a,b)$, $C_0^\infty(\mathbb{R})$, $C_0^\infty(a,b)$
- bounded operator between normed vector spaces
- $\mathcal{L}(V,W)$ for V,W normed vector spaces
- dual space W', for W a normed vector space
- operator norm
- multiplication operator M_f on $L^p(X,d\mu)$, for $f \in L^\infty(X,d\mu)$
- kernel and range of $T \in \mathcal{L}(V,W)$
- linear isometry between normed vector spaces (p. 11)
- left and right shift operators on ℓ^2
- $S^\perp$, for $S \subset \mathcal{H}$ any subset
- direct sum[4] $V \oplus W = X$ of subspaces
- separable Hilbert space
- $w_n \to w$ weakly in $\mathcal{H}$ (p. 27)
- orthonormal (ON) sequence in $\mathcal{H}$
- ON basis in $\mathcal{H}$

Theorems and Lemmas. Know the precise statement of the theorem. Be able to prove if so indicated.

- Heine-Borel theorem for $\mathbb{R}^n$ (Handout)
- extreme value theorem for $\mathbb{R}$-valued functions on compact $K \subset \mathbb{R}^n$ (Handout)
- Fundamental Theorem of Calculus (Handout)
- $\mathcal{L}(V,W)$ is complete in operator norm if W is complete (Theorem 2.10)
- Cauchy-Schwarz (Theorem 2.15)
- parallelogram law (Lemma 2.17; **be able to prove**)
- if $W \subset \mathcal{H}$ is closed then $\mathcal{H} = W \oplus W^\perp$ (Theorem 2.27)
- Riesz lemma (Theorem 2.28)
- $\mathcal{H}$ separable $\iff$ there exists an ON basis of $\mathcal{H}$ (Theorem 2.33)
- Bessel's inequality (Theorem 2.34)

Techniques. Be able to justify and use these calculation/proof techniques.

- To show that $W \subset X$ is a subspace of a complex vector space it suffices to show that $0 \in W$ and that for $v, w \in W$ and $\lambda \in \mathbb{C}$ then $v + \lambda w \in W$.
- If $\mathcal{H}$ is a Hilbert space and $\{e_j\}_{j\in\mathbb{N}}$ is an ON basis of $\mathcal{H}$ then for $v \in \mathcal{H}$ there are coefficients $c_j \in \mathbb{C}$ so that $v = \sum_{j=1}^{\infty} c_j e_j$, and furthermore $c_j = \langle e_j, v\rangle$.
- The triangle inequality $\|x+y\| \le \|x\| + \|y\|$ holds in a normed vector space, and its corollary $\big|\|x\| - \|y\|\big| \le \|x-y\|$ as well.
- The norm in a normed vector space is continuous.
- The inner product in an inner product space is continuous.

[4] We write $V \oplus W = X$ if: *i*) X is a vector space, *ii*) V, W are subspaces of X, *iii*) for each $x \in X$ there exists $v \in V$ and $w \in W$ so that $v + w = x$ (i.e. $V + W = X$), and *iv*) $V \cap W = \{0\}$.