Name: SOLUTIONS

Math 617 Functional Analysis (Bueler) Wednesday 20 March 2024

Midterm Quiz

In-class or proctored. No book, notes, electronics, calculator, internet access, or communication with other people. Precise statements of definitions, theorems, and lemmas are expected. Proofs will be graded generously. If you put work on the blank pages at the end, please clearly label any portions which you would want to be graded. 100 points possible. 65 minutes maximum.

1. Let $(\mathcal{V}, \|\cdot\|)$ be a (complex) normed vector space.

(a) (*5 pts*) Suppose $S \subset \mathcal{V}$. Define what it means for S to be **open**.

def. S is open if for all $x \in S$ there exists $\varepsilon > 0$ so that $B_\varepsilon(x) = \{y \mid \|x-y\| < \varepsilon\} \subset S$.

(b) (*5 pts*) Let $\{v_n\}$ be a sequence in $\mathcal{V}$. Define what it means for this sequence to be **Cauchy**.

def: $\{v_n\}$ is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n, m \geq N$ then $\|v_n - v_m\| < \varepsilon$

2. (*5 pts*) Let $\mathcal{H}$ be a complex Hilbert space. Define $\mathcal{H}'$, the **dual space**.

def: $\mathcal{H}' = \mathcal{L}(\mathcal{H}, \mathbb{C})$

$= \{\ell : \mathcal{H} \to \mathbb{C} \mid \ell \text{ is linear and continuous}\}$

equivalently, bounded

3. (*5 pts*) Suppose $1 \le p \le \infty$. Define $\ell^p = \ell^p(\mathbb{N})$ and its norm. (*Hint. Separate* $p = \infty$.)

def. for $1 \le p < \infty$:

$$\ell^p = \{ a = (a_1, a_2, \dots) \mid \sum_{n=1}^{\infty} |a_n|^p < \infty \}$$

$$\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$$

for $p = \infty$:

$$\ell^\infty = \{ a \mid \exists\, M \ge 0 \text{ s.t. } |a_n| \le M \ \forall n \}$$

$$\|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$$

4. Suppose $\mathcal{H}$ is a complex Hilbert space and $S \subset \mathcal{H}$ is a subset.

(a) (*5 pts*) Define $S^\perp$, the **orthogonal complement** of S.

def. $S^\perp = \{ v \in \mathcal{H} \mid \langle x, v \rangle = 0 \quad \forall x \in S \}$

(b) (*8 pts*) Show that $S^\perp \subset \mathcal{H}$ is a subspace.

proof: Suppose $v, w \in S^\perp$ and $\lambda \in \mathbb{C}$.

Then for $x \in S$:

$$\langle x, v + \lambda w \rangle = \langle x, v \rangle + \lambda \langle x, w \rangle = 0 + \lambda 0 = 0$$

Also

$$\langle x, 0 \rangle = 0.$$

Since $0 \in S^\perp$ and $S^\perp$ is closed under addition and scalar multiplication it is a subspace. $\square$

5. **(a)** (*5 pts*) Define $C_0^\infty(\mathbb{R})$, the vector space of $\mathbb{C}$-valued **smooth functions of compact support**.

def: $C_0^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid f^{(n)}$ exists for all $n \in \mathbb{N}$ and $\exists\, a < b$ real so that $f(x) = 0$ if $x \notin [a,b] \}$

(b) (*8 pts*) Show that if $f, g \in C_0^\infty(\mathbb{R})$ then

$$\int_{\mathbb{R}} f''(x)g(x)\,dx = \int_{\mathbb{R}} f(x)g''(x)\,dx.$$

(*Hint. Carefully do integration by parts.*)

proof: Choose $[a,b]$ so that $f(x) = g(x) = 0$ for all $x \notin [a,b]$. (E.g. choose interval containing the intervals for f, g.) Then

$$\int_{\mathbb{R}} f''(x)g(x)\,dx = \int_a^b f''(x)g(x)\,dx = \Big[f'(x)\underbrace{g(x)}_{g(a)=g(b)=0}\Big]_a^b - \int_a^b f'(x)g'(x)\,dx$$

$$= 0 - \left(\Big[\underbrace{f(x)}_{f(a)=f(b)=0}g'(x)\Big]_a^b - \int_a^b f(x)g''(x)\,dx\right)$$

$$= -0 + \int_a^b f(x)g''(x)\,dx$$

$$= \int_{\mathbb{R}} f(x)g''(x)\,dx.$$

Note all integrands are continuous on $[a,b]$. $\square$

6. (*8 pts*) Suppose V, W are (complex) normed vector spaces, and that $T : V \to W$ is a linear map. Show that if T is bounded then T is continuous.

proof: Let $x \in V$ and $\varepsilon > 0$. Since T is bounded, $\|T\| < \infty$. Let $\delta = \varepsilon / \|T\|$.* Then for $y \in B_\delta(x) \subset V$ we have

$$\|Ty - Tx\|_W = \|T(y-x)\|_W \leq \|T\| \, \|y-x\|$$
$$\leq \|T\| \delta = \varepsilon.$$

(*If $\|T\| = 0$ use $\delta = \varepsilon$.) $\square$

7. (*8 pts*) Let $\mathcal{H}$ be a complex Hilbert space. Suppose that $P \in \mathcal{L}(\mathcal{H})$ satisfies $P^2 = P$ and also that $\langle x, Py \rangle = \langle Px, y \rangle$ for all $x, y \in \mathcal{H}$. Show that if $w = Pu$ for some $u \in \mathcal{H}$, and if $W = \operatorname{range} P$, then $u - w \in W^\perp$.

proof: Let $y \in W$, so $y = Pz$ for $z \in \mathcal{H}$.

Then

$$\langle y, u-w \rangle = \langle Pz, u-w \rangle = \langle z, Pu - Pw \rangle$$
$$= \langle z, Pu - P(Pu) \rangle = \langle z, Pu - P^2 u \rangle$$
$$= \langle \cancel{z}, Pu - Pu \rangle = \langle z, 0 \rangle = 0.$$

Thus $u - w \in W^\perp$ (since $y \in W$ was arbitrary).

$\square$

8. (*8 pts*) State the Riesz lemma. (*No proof is required.*)

Riesz lemma If $\mathcal{H}$ is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and if $\ell \in \mathcal{H}'$, then there is a unique $w \in \mathcal{H}$ so that

$$\ell(u) = \langle w, u \rangle$$

for all $u \in \mathcal{H}$. $\underbrace{\text{Also } \|\ell\| = \|w\|.}_{\text{entirely optional}}$ $\square$

9. (*8 pts*) State the Fundamental Theorem of Calculus. Pay attention to the types of functions to which the Theorem applies. (*No proof is required.*)

FTC Suppose $f \in L^1(a,b)$. Then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous and its derivative exists a.e. Furthermore

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and $\frac{d}{dx} F(x) = f(x)$ a.e.

10. (*11 pts*) Let
$$\phi_k(x) = e^{i2\pi kx}$$
for $k \in \mathbb{Z}$. Then ϕ_k is a continuous, $\mathbb{C}$-valued function on $[0,1]$, so $\phi_k \in L^2(0,1)$. (*There is no need to prove this.*) Show that
$$\{\phi_j(x)\phi_k(y)\}_{j,k\in\mathbb{Z}}$$
is an orthonormal set on $L^2(\Omega)$, where $\Omega = (0,1)^2$.

proof. Let $j, k, r, s \in \mathbb{Z}$. Note

$$\int_0^1 \overline{\phi_j(x)}\,\phi_k(x)\,dx = \int_0^1 e^{i2\pi(k-j)x}\,dx = \left.\begin{cases} 1, & j=k \\ 0, & \text{otherwise} \end{cases}\right\} = \delta_{jk}$$

since $\left[e^{i2\pi(k-j)x}/2\pi(k-j)\right]_0^1 = 0$ if $k-j \neq 0$.

Thus (* = by Fubini's theorem, but you don't need to say it)

$$\int_\Omega \overline{\phi_j(x)\phi_k(y)}\,\phi_r(x)\phi_s(y)\,dx\,dy$$

$$\overset{*}{=} \int_0^1\int_0^1 \overline{\phi_j(x)}\,\phi_r(x)\,\overline{\phi_k(y)}\,\phi_s(y)\,dx\,dy$$

$$\overset{*}{=} \left(\int_0^1 \overline{\phi_j(x)}\,\phi_r(x)\,dx\right)\left(\int_0^1 \overline{\phi_k(y)}\,\phi_s(y)\,dy\right)$$

$$= \delta_{jr}\,\delta_{ks}.$$

So if $\Phi_{jk}(x,y) = \phi_j(x)\phi_k(y)$ then

$$\langle \Phi_{jk}, \Phi_{rs}\rangle_{L^2(\Omega)} = \begin{cases} 1, & j=r \text{ and } k=s \\ 0, & \text{otherwise,} \end{cases}$$

so $\{\Phi_{jk}\}_{j,k\in\mathbb{Z}}$ is an ON set. $\square$

11. (*11 pts*) Let $\mathcal{H} = \ell^2$ and suppose $R \in \mathcal{L}(\mathcal{H})$ is the right-shift operator

$$R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

(*There is no need to prove that* $R \in \mathcal{L}(\mathcal{H})$.) Show that R has no eigenvalues.

<u>proof</u> Suppose $Ra = \lambda a$ for $a \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$ then

$$Ra = (0, a_1, a_2, a_3, \dots) = (0, 0, 0, \dots)$$

so $a_1 = a_2 = a_3 = \dots = 0$, so $a = 0$. Thus $\lambda = 0$ is not an eigenvalue.

If $\lambda \neq 0$ then

$$Ra = (0, a_1, a_2, a_3, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \dots) = \lambda a.$$

But then $\lambda a_1 = 0$ so $a_1 = 0$. And $\lambda a_2 = a_1$ so $a_2 = a_1/\lambda = 0/\lambda = 0$. Continuing by induction, $\lambda a_{n+1} = a_n$ and $a_n = 0$ so $a_{n+1} = a_n/\lambda = 0/\lambda = 0$. In conclusion $a_n = 0$ for all $n \in \mathbb{N}$ so $a = 0$, so λ is not an eigenvalue. $\square$

Extra Credit. (*4 pts*) The ON set $\{\phi_j(x)\phi_k(y)\}$ in problem **10**, for $j, k \in \mathbb{Z}$, is actually an ON *basis* of $L^2(\Omega)$ where $\Omega = (0,1)^2$. Furthermore this basis diagonalizes the Laplacian operator

$$Lu = u_{xx} + u_{yy}.$$

We will see that L is an unbounded operator on $L^2(\Omega)$. (*There is no need to prove any of the previous statements.*) Find all the eigenvalues of L.

Since $\{\Phi_{jk}(x,y)\} = \{\phi_j(x)\phi_k(y)\}$ diagonalizes L,

we can simply apply L:

$$L\Phi_{jk} = (\Phi_{jk})_{xx} + (\Phi_{jk})_{yy}$$

$$= (i2\pi j)^2 \Phi_{jk} + (i2\pi k)^2 \Phi_{jk}$$

$$= -4\pi^2(j^2+k^2)\Phi_{jk}$$

so

$$\underbrace{\sigma(L)}_{=\text{ spectrum of } L \text{ (which is the set of eigenvalues in this case)}} = \{-4\pi^2(j^2+k^2)\}_{j,k\in\mathbb{Z}}.$$

(Note that many eigenvalues have large multiplicity. E.g. $\lambda = -16\pi^2$ is for $\Phi_{-2,0}, \Phi_{2,0}, \Phi_{0,-2}, \Phi_{0,2}$; multiplicity $=4$.)