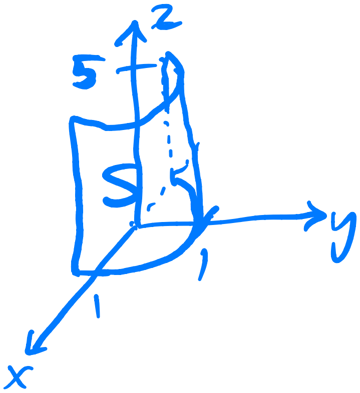
SOLUTIONS

Worksheet: Surface integrals

1. Sketch the parameterized surface S given by $\mathbf{r}(u, v) = \langle \cos v, \sin v, u\rangle$ for $0 \le u \le 5$ and $0 \le v \le \pi$. Then compute the surface integral

$$\iint_S z\, dS = \int_0^{\pi} \int_0^5 \underbrace{u}_{= z} \cdot \underbrace{1}_{= \|\vec{r}_u \times \vec{r}_v\|}\, du\, dv$$



$$= \pi \int_0^5 u\, du$$

$$= \pi \left[\frac{u^2}{2}\right]_0^5 = \frac{25\pi}{2}$$

$$\vec{r}_u = \langle 0, 0, 1\rangle$$

$$\vec{r}_v = \langle -\sin v, \cos v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -\sin v & \cos v & 0 \end{vmatrix}$$

$$= \langle -\cos v, -\sin v, 0\rangle$$

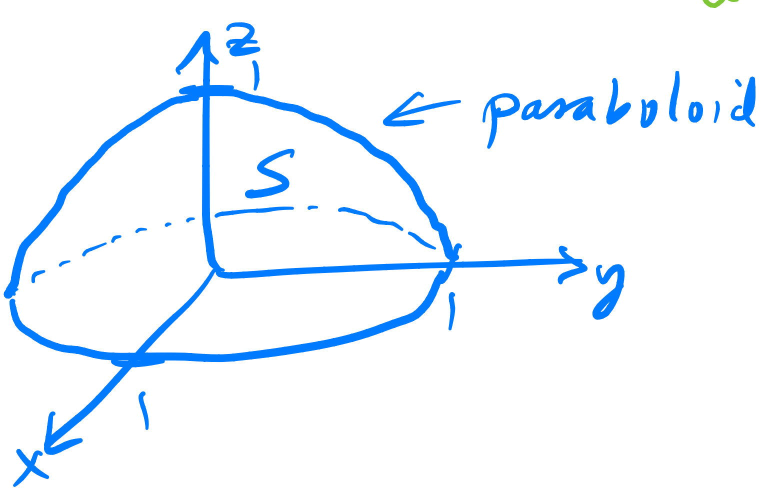
$$\|\vec{r}_u \times \vec{r}_v\| = 1$$

Compare: $\left[A_S = \iint_S 1\, dS = 5\pi \right]$

2. Let S be the part of the graph (surface) $z = 1 - x^2 - y^2$ for which $z \ge 0$. Parameterize this surface. Set-up, and compute, a surface integral for its area.

$$\vec{r}(u, v) = \langle u\cos v, u \sin v, 1 - u^2\rangle \qquad 0 \le u \le 1, \quad 0 \le v \le 2\pi$$

so: $x^2 + y^2 = u^2\cos^2 v + u^2 \sin^2 v$
$= u^2 \quad (\therefore z = 1 - x^2 - y^2)$



$$A_S = \iint_S 1\, dS$$

continued →

2. cont.

$$\vec{t}_u = \langle \cos v, \sin v, -2u \rangle$$

$$\vec{t}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= (0 + 2u^2\cos v)\hat{i} - (0 - 2u^2\sin v)\hat{j} + (u\cos^2 v + u\sin^2 v)\hat{k}$$

$$= u\langle 2u\cos v, 2u\sin v, 1 \rangle$$

$$\|\vec{t}_u \times \vec{t}_v\| = u\sqrt{4u^2\cos^2 v + 4u^2\sin^2 v + 1}$$

$$= u\sqrt{4u^2+1}$$

So:

$$A_s = \iint_S 1\,dS = \int_0^{2\pi}\int_0^1 1 \cdot u\sqrt{4u^2+1}\,du\,dv$$

$$= 2\pi \int_0^1 u\sqrt{4u^2+1}\,du$$

$w = 4u^2+1$

$dw = 8u\,du$

$\frac{dw}{8} = u\,du$

$$= 2\pi \int_1^5 \sqrt{w}\,\frac{dw}{8}$$

$$= \frac{\pi}{4}\left[\frac{2}{3}w^{3/2}\right]_1^5 = \frac{\pi}{6}\left(5^{3/2} - 1\right)$$