Math 253: Quiz 9 SOLUTIONS Thursday 13 April, 2023

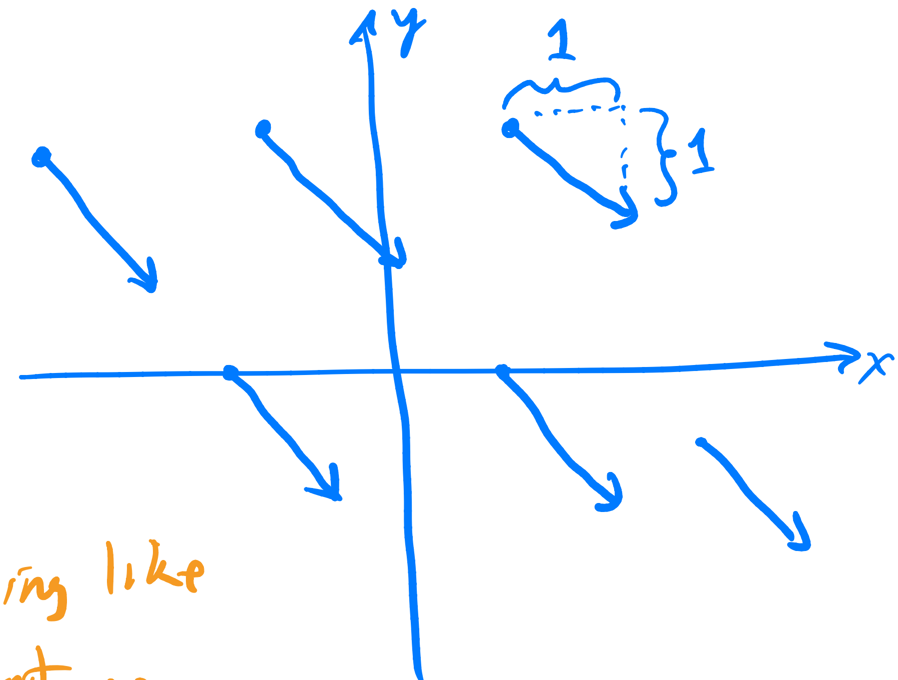
Name: SOLUTIONS / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

**1. [4 points]** Sketch the vector field $\mathbf{F}(x,y) = \mathbf{i} - \mathbf{j}$. Describe the vector field in a sentence. (*Draw at least 6 vectors in your sketch. Regarding the sentence, what is the length of the vectors, and their directions?*)

The vector field has constant length $\sqrt{2}$ and always points south east.

↑ or something like that...




**2. [3 points]** Compute the gradient vector field of $f(x,y,z) = xy - yz + z^3$.

$$\nabla f(x) = \langle f_x, f_y, f_z \rangle$$

$$= \langle y, x-z, -y+3z^2 \rangle$$

**3. [6 points]** Find the work done by the force field $\mathbf{F}(x,y) = -2y\mathbf{i} + 2x\mathbf{j} + (x+y)\mathbf{k}$ in moving an object along the curve $\mathbf{r}(t) = \underbrace{\cos(t)}_{x(t)}\mathbf{i} + \underbrace{\sin(t)}_{y(t)}\mathbf{j} \underbrace{-3}_{z(t)}\mathbf{k}$, where $0 \le t \le \pi$.

$$W = \int_C \vec{F} \cdot \underbrace{d\vec{r}}_{= \frac{d\vec{r}}{dt}\,dt} = \int_0^\pi \langle -2y(t),\ 2x(t),\ x(t)+y(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt$$

$$= \int_0^\pi +2\sin(t)\sin(t) + 2\cos(t)\cos(t) + \cancel{(\ldots)}0 \, dt$$

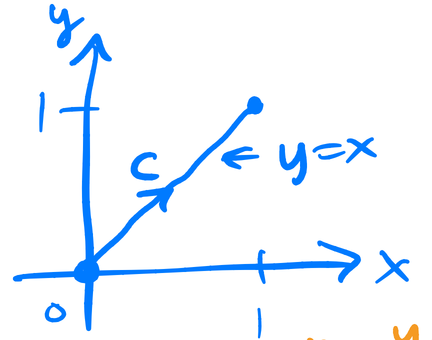
$$= 2\int_0^\pi \sin^2 t + \cos^2 t \, dt = 2\int_0^\pi dt$$

$$= 2\pi$$

**4. [3 points]** Find $\mathbf{T}(t)$, the unit tangent vector field, for the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1+4t^2+9t^4}}$$

**5. [6 points]** Evaluate the line integral $\int_C f(x,y)\,ds$ of the scalar function $f(x,y) = x+y$ along the straight-line path connecting the origin to the point $(1,1)$.




$\int_C f(x,y)\,ds$ ← remember: $\frac{ds}{dt} = \|\vec{r}\,'(t)\|$

$= \int_{t=0}^{t=1} f(x(t),y(t))\,\frac{ds}{dt}\,dt$

$C: \vec{r}(t) = \langle \underset{x}{t}, \underset{y}{t} \rangle$

$0 \le t \le 1$

$= \int_0^1 (t+t)\sqrt{2}\,dt$

$\|\vec{r}\,'(t)\| = \|\langle 1, 1\rangle\|$

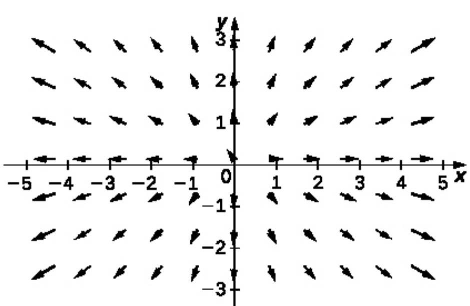
$= \sqrt{1^2+1^2} = \sqrt{2}$

$= 2\sqrt{2}\int_0^1 t\,dt$

$= 2\sqrt{2}\cdot\left[\frac{t^2}{2}\right]_0^1 = 2\sqrt{2}\cdot\frac{1}{2}$

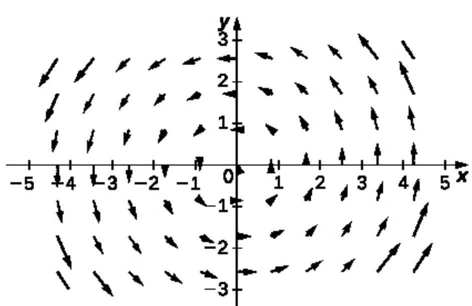
$= \sqrt{2}$

**6. [3 points]** Consider the vector fields $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, $\mathbf{G} = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{H} = x\mathbf{i} - y\mathbf{j}$. Match $\mathbf{F}, \mathbf{G}, \mathbf{H}$ to their graphs below. In particular, write the variable name below the matching graph.



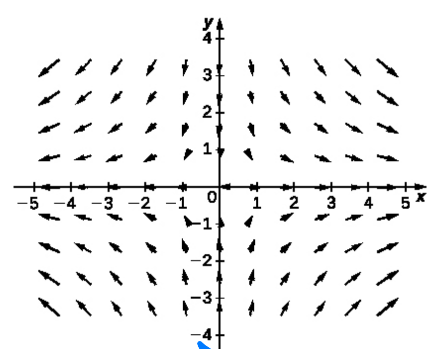
$\vec{G}$ ______

↑ same as $\vec{r} = \langle x, y \rangle$



$\vec{F}$ ______

↑ note $\vec{F}\cdot\vec{r} = 0$ (so tangent to circles)



$\vec{H}$ ______

↑ check along axes?

**Extra Credit. [1 point]** I claim $\mathbf{F}(x,y) = \langle e^x, y+e^x\rangle$ is a conservative vector field. Find a potential $f(x,y)$.

not true!

Sorry!

$\vec{F}$ is not conservative because

$P = e^x$, $Q = y + e^x$

and $P_y = 0 \neq Q_x = e^x$.

I suppose I meant "$\vec{F} = \langle ye^x, y + e^x\rangle$".

Then $P_y = e^x = Q_x$. Then $f_x = ye^x$ so

$f(x,y) = ye^x + g(y)$. So $e^x + g'(y) = f_y = y + e^x$.

EXTRA SPACE FOR ANSWERS

Then $g'(y) = y$

So $g(y) = \frac{y^2}{2} + C$

So $f(x,y) = ye^x + \frac{y^2}{2} + C$