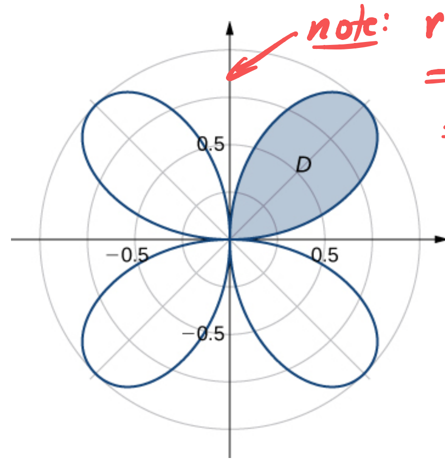
**Math 253: Quiz 8** SOLUTIONS **Thursday 30 March, 2023**

**Name:** ______________________________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

**1. [5 points]** Find the area of one leaf of the rose $r = \sin(2\theta)$, which is shown in the figure.




note: $r(\pi/2) = \sin(2\pi/2) = 0$

$$A = \iint_D 1\,dA$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\sin(2\theta)} 1\, r\,dr\,d\theta$$

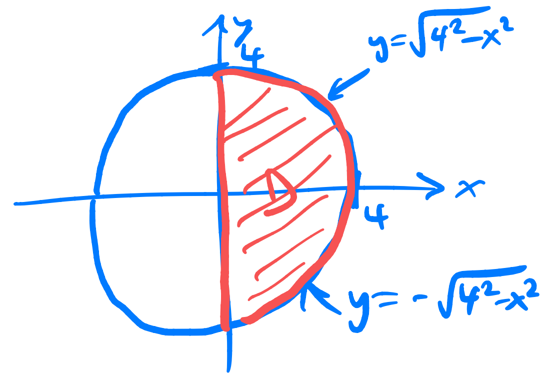
$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sin(2\theta)} d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1-\cos(4\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\pi/2 - 0 - (0-0)\right] = \pi/8$$

**2. [5 points]** Convert the integral to polar coordinates. There is no need to evaluate the integral! (**Hint.** Sketch the region of integration, which tells you the limits on the $r, \theta$ integrals.)

$$\int_0^4 \int_{-\sqrt{16-x^2}}^{+\sqrt{16-x^2}} \arctan(x^2+y^2)\,dy\,dx =$$




$$= \iint_D \arctan(r^2)\, r\,dr\,d\theta$$

$$= \boxed{\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{4} \arctan(r^2)\,r\,dr\,d\theta} = \pi\int_0^4 \arctan(r^2)\,r\,dr$$

easy to do $\theta$ integral, if you want

**3. [5 points]** Using mathematically-correct steps, show that:

$$\int_a^b \int_c^d \int_e^f F'(x)G'(y)H'(z)\,dz\,dy\,dx = [F(b)-F(a)]\,[G(d)-G(c)]\,[H(f)-H(e)]$$

(**Hint.** Start on the left. What terms can be moved out of the inner integrals? What do you know about the integral of a derivative?)

$$\int_a^b \int_c^d \int_e^f F'(x)\,G'(y)\,H'(z)\,dz\,dy\,dx$$

$$= \int_a^b \int_c^d F'(x)\,G'(y)\left(\int_e^f H'(z)\,dz\right)dy\,dx$$

$$= \left(\int_a^b F'(x)\,dx\right)\left(\int_c^d G'(y)\,dy\right)\left(\int_e^f H'(z)\,dz\right)$$

$$= [F(b)-F(a)]\,[G(d)-G(c)]\,[H(f)-H(e)]$$

FTC 3 times!

**4. [5 points]** Assume $B = \{(x,y,z)\,|\,1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$. Evaluate the triple integral:

$$\iiint_B xy\,dV = \int_{x=1}^{2}\int_{y=0}^{2}\int_{z=1}^{3} x\,y\,dz\,dy\,dx$$

$$= \int_1^2 \int_0^2 xy\left(\int_1^3 dz\right)dy\,dx = 2\int_1^2\int_0^2 x\,y\,dy\,dx$$

$$= 2\int_1^2 x\left[\frac{y^2}{2}\right]_0^2 dx = 2\cdot 2\int_1^2 x\,dx$$

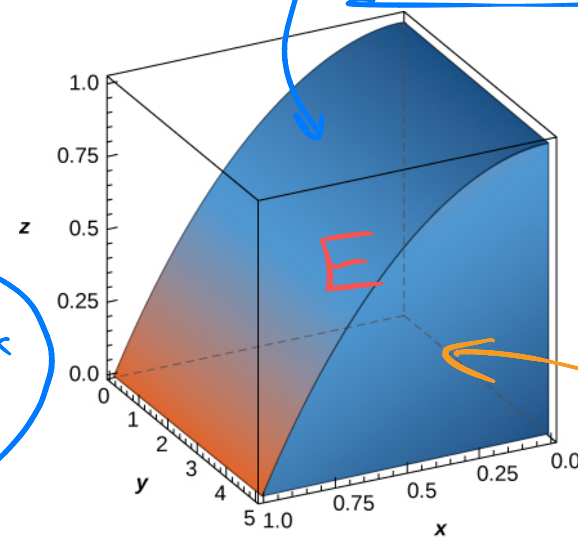
$$= 4\left[\frac{x^2}{2}\right]_1^2 = \frac{4}{2}\left(2^2-1^2\right) = 2(3) = 6$$

**5.** **[5 points]** A solid object is shown. It is the set in the first octant which bounded by $z = 1 - x^2$ and the plane $y = 5$. Supposing its density is $\rho(x,y,z) = 1 + x + y$, **completely set up** a triple integral to find its total mass.

$$M = \iiint_E 1+x+y \, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{5} \int_{z=0}^{1-x^2} 1+x+y \, dz \, dy \, dx$$

upper surface




$y=5$ is front face

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EXTRA SPACE FOR ANSWERS

**Extra Credit. [1 point]** Compute and fully simplify the integral in problem **5**.

$$M = \int_{x=0}^{1}\int_{y=0}^{5}\int_{z=0}^{1-x^2} 1+x+y \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^5 (1+x+y)\left[z\right]_0^{1-x^2} dy\, dx$$

$$= \int_0^1\int_0^5 (1+x+y)(1-x^2)\, dy\, dx = \int_0^1\int_0^5 1-x^2+x-x^3+y(1-x^2) dy\, dx$$

$$= \int_0^1 \left[(1-x^2+x-x^3)y + (1-x^2)\frac{y^2}{2}\right]_0^5 dx$$

$$= \int_0^1 5(1-x^2+x-x^3) + \frac{25}{2}(1-x^2)\, dx$$

$$= \left[5\left(x-\frac{x^3}{3}+\frac{x^2}{2}-\frac{x^4}{4}\right)+\frac{25}{2}\left(x-\frac{x^3}{3}\right)\right]_0^1$$

$$= 5\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}\right)+\frac{25}{2}\left(1-\frac{1}{3}\right)$$

$$= 5\left(\frac{12-4+6-3}{12}\right)+\frac{25}{\cancel{2}}\cdot\frac{\cancel{2}}{3} = 5\left(\frac{11}{12}\right)+\frac{25}{3}$$

$$= \frac{55+25\cdot 4}{12} = \frac{155}{12}$$