Math 253: Quiz 7 SOLUTIONS **23 March, 2023**

Name: ______________________________ / **25**

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. **[5 points]** Evaluate the iterated (double) integral:

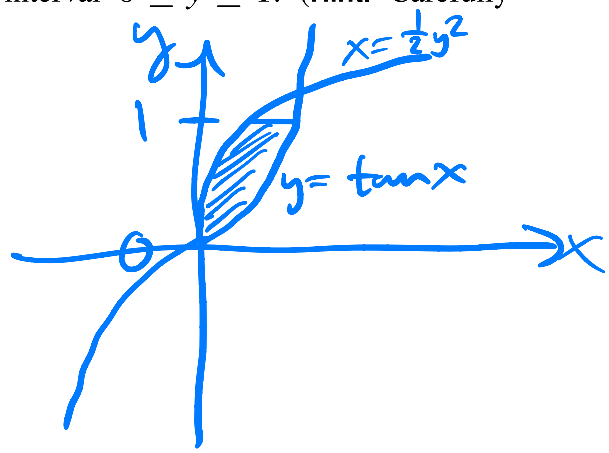
$$\int_1^2\int_0^9 \frac{\sqrt{y}}{x^2}\,dy\,dx = \int_1^2 \frac{1}{x^2}\left[\frac{2}{3}y^{3/2}\right]_0^9 dx = \int_1^2 x^{-2}\left(\frac{2}{3}\cdot 3^3\right)dx$$

$$= 18\int_1^2 x^{-2}\,dx = 18\left[-x^{-1}\right]_1^2 = 18\left[-\frac{1}{2}+1\right]$$

$$= 9$$

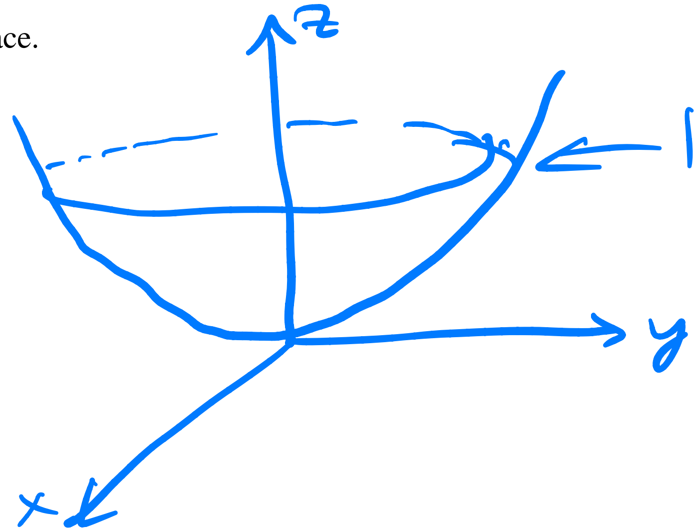
2. **[5 points]** Set up, but do not evaluate, a double integral to compute the **area** of the region between the graph $x = \frac{1}{2}y^2$ and the graph $y = \tan x$, over the interval $0 \le y \le 1$. (**Hint.** Carefully sketch the region.)

$$A = \int_0^1 \int_{x=\frac{1}{2}y^2}^{x=\arctan y} 1\,dx\,dy$$



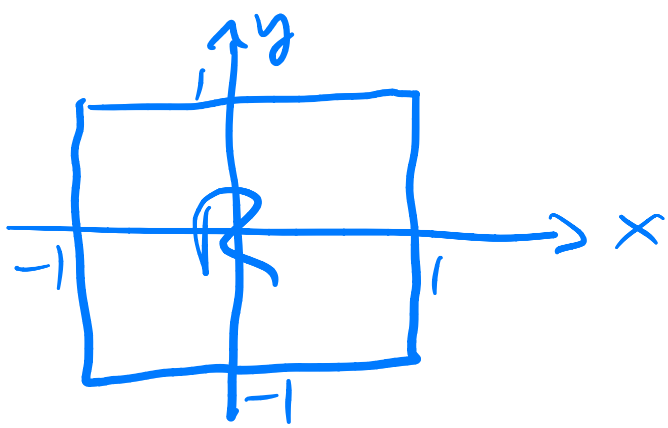
3. [8 points] Consider the surface $z = x^2 + y^2$ and the region $R = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le 1\}$.

a) Sketch the surface.



level curves are circles: $c = x^2 + y^2$

b) Sketch the region R.



c) Find the volume under the surface and above the region R.

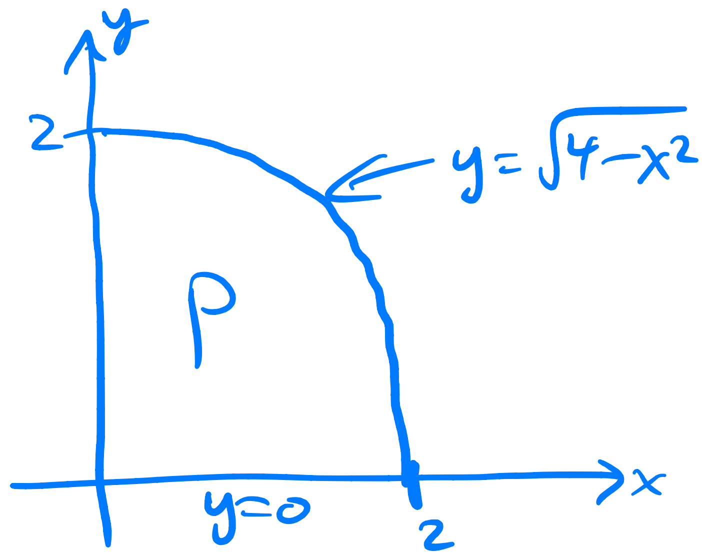
using symmetry is optimal

$$V = \int_{-1}^{1}\int_{-1}^{1} x^2 + y^2 \, dx\, dy = 4\int_0^1\int_0^1 x^2 + y^2 \, dx\, dy$$

$$= 4\int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_0^1 dy = 4\int_0^1 \left(\frac{1}{3} + y^2\right) dy$$

$$= 4\left[\frac{1}{3}y + \frac{y^3}{3}\right]_0^1 = 4\cdot\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{8}{3}$$

4. [7 points] Suppose we have a plate of metal in the shape of a quarter circle, bounded by the curves $x = 0$, $y = 0$, and $x^2 + y^2 = 4$. Find the average temperature of the plate if the temperature is given by the function $T(x, y) = 10x$.



$$A_P = \frac{\pi \cdot 2^2}{4} = \pi$$

$$T_{av} = \frac{1}{\pi} \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} 10x \, dy \, dx$$

$$= \frac{1}{\pi} \int_0^2 10x \left[y \right]_0^{\sqrt{4-x^2}} dx$$

$$= \frac{1}{\pi} \int_0^2 10x \sqrt{4-x^2} \, dx$$

$u = 4 - x^2$

$\frac{du}{-2} = x\,dx$

$$= \frac{10}{\pi} \int_4^0 \sqrt{u} \, \frac{du}{-2} = \frac{5}{\pi} \int_0^4 u^{1/2} \, du$$

$$= \frac{5}{\pi} \left[\frac{2}{3} u^{3/2} \right]_0^4 = \frac{5}{\pi} \cdot \frac{2}{3} \cdot 2^3 = \frac{80}{3\pi}$$

Extra Credit. [1 point] Suppose $f(x)$ is any continuous function and consider the domain

$$D = \{(x,y) \,|\, a \leq x \leq b,\, f(x) \leq y \leq f(x)+1\}.$$

Use a double integral to show that the area of D is just $A_D = b - a$.

$$A_D = \int_a^b \int_{y=f(x)}^{y=f(x)+1} 1 \, dy\, dx$$

$$= \int_a^b [y]_{f(x)}^{f(x)+1} \, dx = \int_a^b f(x)+1 - f(x) \, dx$$

$$= \int_a^b 1 \, dx = b - a$$

EXTRA SPACE FOR ANSWERS