Math 253: Quiz 6 SOLUTIONS **9 March, 2023**

Name: ____________________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [7 points] Consider the function $f(x,y) = e^x \cos y$.

a) Compute the gradient $\nabla f(x,y)$.

$$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$$

b) Compute the directional derivative of f at the point $P\left(1, \frac{\pi}{2}\right)$ in the direction $\mathbf{v} = -\mathbf{i}$.

$$D_{\vec{v}} f(1, \pi/2) = \langle e \cdot 0, -e \cdot 1 \rangle \cdot \langle -1, 0 \rangle$$

$$= \langle 0, -e \rangle \cdot \langle -1, 0 \rangle$$

$$= 0$$

2. [5 points] Find the maximum rate of change of $f(x,y) = x\ln y$ at the point $(2,1)$, and the direction in which it occurs.

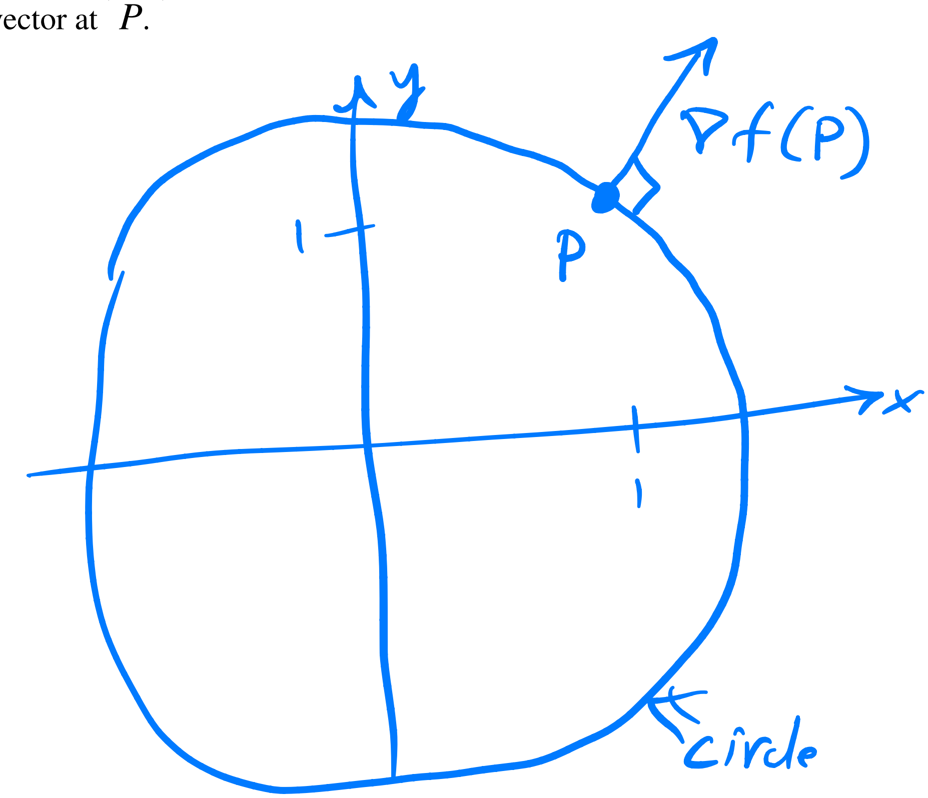
$$\nabla f(x,y) = \langle 1 \cdot \ln y,\ x \cdot \tfrac{1}{y} \rangle$$

$$\vec{v} = \nabla f(1,2) = \langle 0,\ 2 \cdot \tfrac{1}{1} \rangle = \langle 0, 2 \rangle$$

max rate of change $= \|\vec{v}\| = 2$

direction $= \dfrac{\vec{v}}{\|\vec{v}\|} = \langle 0, 1 \rangle = \hat{j}$

3. [5 points] Sketch the level curve of $f(x,y) = 3x^2 + 3y^2$ which passes through the point $P(1,1)$, and draw the gradient vector at P.



$3x^2 + 3y^2 = 6$

4. [8 points] Consider the function $f(x,y) = x^3 + y^3 - 3x - 12y - 2$.

a) Find all the critical points.

$f_x = 3x^2 - 3 = 0 \iff x^2 = 1$

$f_y = 3y^2 - 12 = 0 \iff y^2 = 4$

$(1, 2)$
$(1, -2)$
$(-1, 2)$
$(-1, -2)$

4 critical points

b) For each critical point, use the second derivative test to determine if it is a local minimum, local maximum, or saddle point.

$D = f_{xx} f_{yy} - f_{xy}^2 = 6x \cdot 6y - 0^2$

$= 36xy$

point	D	f_{xx}	type
$(1,2)$	$+$	$+$	local min.
$(1,-2)$	$-$	$+$	saddle
$(-1,2)$	$-$	$-$	saddle
$(-1,-2)$	$+$	$-$	local. max

Extra Credit. [1 point] Show that the gradient of a function $f(x,y)$ is orthogonal to its level curves. (**Hint.** Write down the equation for a level curve. Suppose the level curve is parameterized. Take derivatives of both sides of the equation.)

$$f(x,y) = C$$

parameterized: $$f(\overbrace{x(t), y(t)}^{\text{level curve is } \vec{r}(t) = \langle x(t), y(t)\rangle}) = C$$

$\frac{d}{dt}$: $$\frac{\partial f}{\partial x}(x(t),y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t),y(t))\, y'(t) = 0$$

$$\nabla f(x(t),y(t)) \cdot \vec{r}\,'(t) = 0$$

so gradient is orthogonal to tangent of curve

EXTRA SPACE FOR ANSWERS