Math 253: Quiz 4 SOLUTIONS **16 February, 2023**

Name: ______________________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [4 points] Find the arc length of the vector-valued function $\mathbf{r}(t) = -t\,\mathbf{i} + 4t\,\mathbf{j} + 3t\,\mathbf{k}$ over $[0,1]$. (*Hint.* You can do this either way, with or without an integral.)

without integral:

curve is a straight line

$\vec{r}(0) = \langle 0,0,0 \rangle, \quad \vec{r}(1) = \langle -1,4,3 \rangle$

$s = \left(\text{distance between points}\right) = \sqrt{1^2+4^2+3^2} = \sqrt{26}$

with integral:

$$s = \int_0^1 \|\vec{r}\,'(t)\|\,dt = \int_0^1 \|\langle -1,4,3 \rangle\|\,dt$$

$$= \int_0^1 \sqrt{1^2+4^2+3^2}\,dt = \sqrt{26}\int_0^1 dt = \sqrt{26}$$

2. [4 points] Compute the arc-length function $s(t)$ for the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t = 0$.

$$s(t) = \int_0^t \|\vec{r}\,'(u)\|\,du = \int_0^t \|\langle -\sin u, \cos u, 1 \rangle\|\,du$$

$$= \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1}\,du = \int_0^t \sqrt{2}\,du$$

$$= \sqrt{2}\int_0^t du = \sqrt{2}\,[u]_0^t = \sqrt{2}\,t$$

3. [5 points] Explain in 2 or 3 complete sentences what the following definition of curvature, given in section 3.3, is saying:

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

(*Hint.* What are the objects on the right side? Use the phrase "rate of change" where appropriate. And what is the curvature geometrically?)

If $\vec{r}(s)$ is a curve parameterized by arclength, with unit tangent $\vec{T}(s)$, then the curvature $\kappa(s)$ is the magnitude of the rate of change of $\vec{T}(s)$. The number $\kappa(s)$ is $\frac{1}{R}$ where R is the radius of the osculating circle at $\vec{r}(s)$.

4. [4 points] Find the level surface of the three-variable function $w(x,y,z) = x^2+y^2+z^2$ at $c = 36$. Describe this surface in a complete sentence.

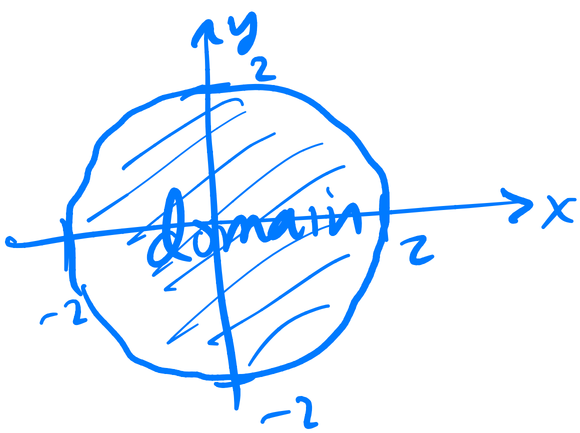
level surface: $x^2+y^2+z^2=36$

This surface is the sphere of radius 6 centered at the origin.

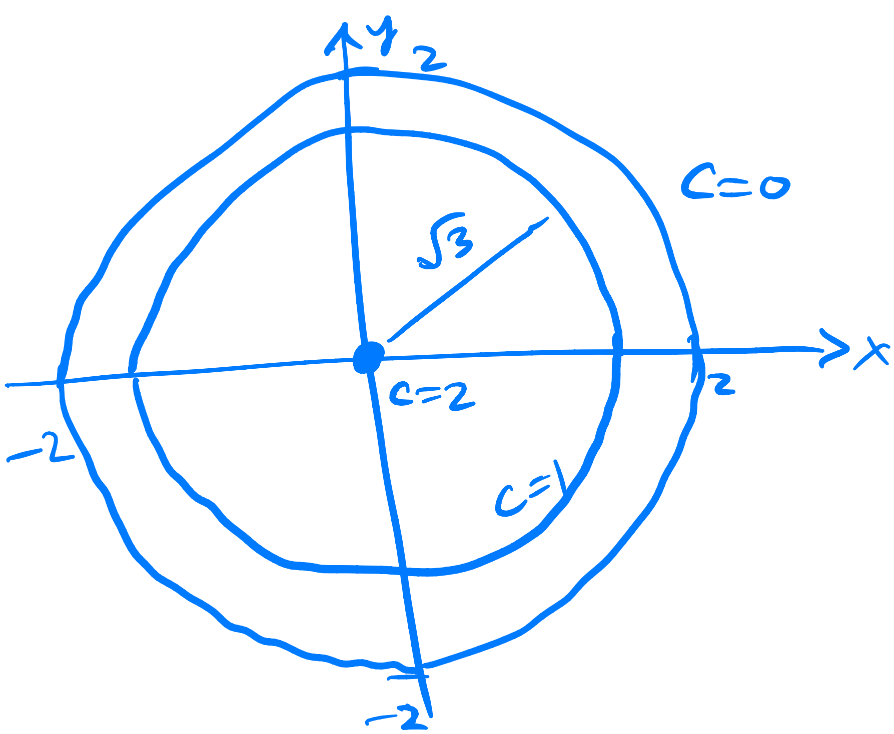
5. [4 points] Find and sketch the domain of the function $f(x,y) = \sqrt{4 - x^2 - y^2}$.

$$(\text{domain}) = \{(x,y) \mid 4 - x^2 - y^2 \geq 0\}$$

$$= \{(x,y) \mid x^2 + y^2 \leq 4\}$$



6. [4 points] Visualize the same function $f(x,y) = \sqrt{4 - x^2 - y^2}$ by finding and sketching at least three level curves. Label the curves with their function value, that is, their "c" value.



$c = 0$: $x^2 + y^2 = 4$

$c = 1$: $x^2 + y^2 = 3$

$c = 2$: $x^2 + y^2 = 0$

Extra Credit. [1 point] Given the definition $\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\|$, show that $\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ for a vector-valued function $\mathbf{r}(t)$.

$$\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| \overset{*}{=} \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

regarding the denominator in $*$:

$s(t) = \int_a^t \|\vec{r}'(u)\| \, du$ so $\frac{ds}{dt} = \|\vec{r}'(t)\|$

by FTC.

EXTRA SPACE FOR ANSWERS