Math 253: Quiz 3 **9 February, 2023**

Name: SOLUTIONS

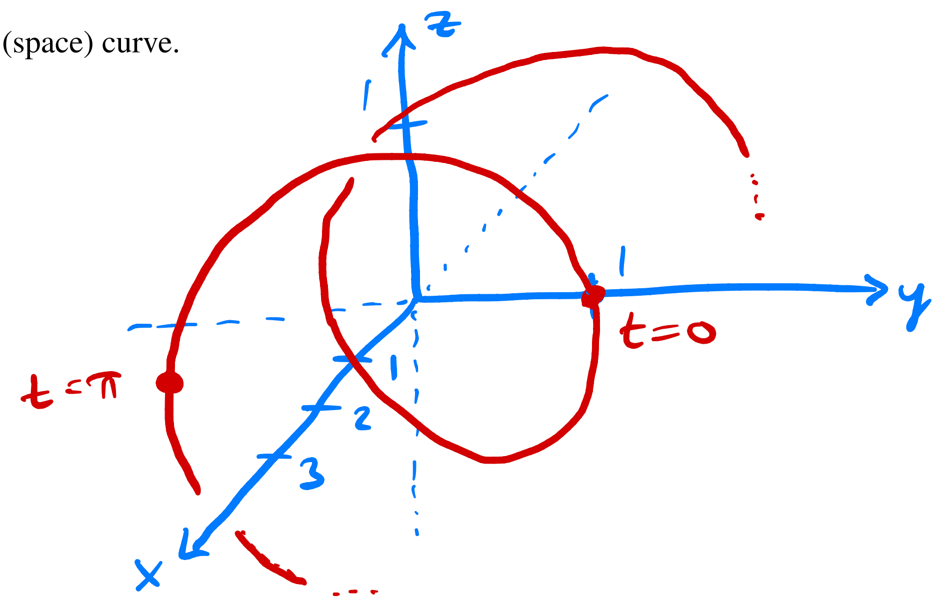
/ 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [5 points] Suppose $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$.

a) Sketch this three-dimensional (space) curve.

helix aligned with x-axis



b) Is $\mathbf{r}(t)$ continuous at $t=\pi$? If so, explain why in a few words, and state as an equation.

Yes. $\vec{r}(\pi) = \lim_{t\to\pi} \vec{r}(t) = \langle \pi, -1, 0\rangle$.

(The limit at $t=\pi$ matches the value at $t=\pi$.)

2. [4 points] Consider the vector-valued function $\mathbf{r}(t) = t^2\,\mathbf{i} + \sqrt{t-3}\,\mathbf{j} + \frac{5}{2t-1}\,\mathbf{k}$.

a) What is the domain of $\mathbf{r}(t)$?

$t \neq \frac{1}{2}$

$\{t : t \geq 3 \text{ and } 2t-1 \neq 0\}$

$= \{t : t \geq 3\} = [3, \infty)$

b) Compute $\lim_{t\to 3^+} \mathbf{r}(t)$.

$\lim_{t\to 3^+} \vec{r}(t) = 9\hat{\imath} + 0\hat{\jmath} + \frac{5}{6-1}\hat{k} = \langle 9, 0, 1\rangle$

$\lim_{t\to 3^+} \sqrt{t-3} = \lim_{x\to 0^+} \sqrt{x} = 0$

3. [8 points] Suppose that a moving particle has position function $\mathbf{r}(t) = \langle e^{-t}, t, te^{-t} \rangle$.

a) What is the velocity $\mathbf{v}(t)$ at time $t = 1$?

$$\vec{v}(t) = \langle -e^{-t}, 1, e^{-t} + t(-e^{-t}) \rangle$$
$$= \langle -e^{-t}, 1, e^{-t}(1-t) \rangle$$
$$\vec{v}(1) = \langle -\tfrac{1}{e}, 1, \tfrac{1}{e} \cdot 0 \rangle = \boxed{\langle -\tfrac{1}{e}, 1, 0 \rangle}$$

b) Calculate the tangent line to the curve $\mathbf{r}(t)$ at $t = 1$.

$$\vec{r}(1) = \langle \tfrac{1}{e}, 1, \tfrac{1}{e} \rangle$$

line:
$$\vec{u}(t) = \underbrace{\vec{r}(1)}_{\vec{r}_0} + t \underbrace{\vec{v}(1)}_{\vec{v}}$$
$$= \langle \tfrac{1}{e}, 1, \tfrac{1}{e} \rangle + t \langle -\tfrac{1}{e}, 1, 0 \rangle$$
$$= \boxed{\langle \tfrac{1}{e}(1-t), 1+t, \tfrac{1}{e} \rangle}$$

integration-by-parts

$u = t \quad v = -e^{-t}$

$du = dt \quad dv = e^{-t}dt$

c) Compute $\int_0^1 \mathbf{r}(t)\, dt =$

$$\int_0^1 \vec{r}(t)\,dt = \langle \int_0^1 e^{-t}\,dt, \int_0^1 t\,dt, \int_0^1 te^{-t}\,dt \rangle$$
$$= \langle \left[-e^{-t}\right]_0^1, \left[\tfrac{t^2}{2}\right]_0^1, \left[t(-e^{-t})\right]_0^1 - \int_0^1 (-e^{-t})\,dt \rangle$$
$$= \langle -e^{-1} + 1, \tfrac{1}{2} - 0, (-e^{-1} + 0) + \int_0^1 e^{-t}\,dt \rangle$$
$$= \langle 1 - \tfrac{1}{e}, \tfrac{1}{2}, \tfrac{-1}{e} - \left[e^{-t}\right]_0^1 \rangle = \boxed{\langle 1 - \tfrac{1}{e}, \tfrac{1}{2}, 1 - \tfrac{2}{e} \rangle}$$

4. [8 points] The vector-valued function $\mathbf{r}(t) = a\sin(\omega t)\,\mathbf{i} + a\cos(\omega t)\,\mathbf{j}$, for constants $a > 0$ and $\omega > 0$, describes uniform circular motion.

a) Show that the velocity $\mathbf{v}(t)$ at any time is orthogonal to the position $\mathbf{r}(t)$.

$$\vec{v}(t) = a\omega\cos(\omega t)\hat{i} - a\omega\sin(\omega t)\hat{j}$$

$$\vec{v}(t)\cdot\vec{r}(t) = \langle a\omega\cos(\omega t), -a\omega\sin(\omega t)\rangle \cdot \langle a\sin(\omega t), a\cos(\omega t)\rangle$$

$$= a^2\omega\left(\cos(\omega t)\sin(\omega t) - \sin(\omega t)\cos(\omega t)\right)$$

$$= a^2\omega(0) = 0 \quad \therefore \text{ orthogonal}$$

b) Show that the acceleration $\mathbf{a}(t)$ at any time is orthogonal to the velocity $\mathbf{v}(t)$.

$$\vec{a}(t) = a\omega^2\langle -\sin(\omega t), -\cos(\omega t)\rangle$$

$$= -\omega^2\vec{r}(t)$$

from a)

$$\vec{a}(t)\cdot\vec{v}(t) = -\omega^2\,\vec{r}(t)\cdot\vec{v}(t) = 0$$

$$\therefore \text{ orthogonal}$$

<u>or</u> you can compute $\vec{a}(t)\cdot\vec{v}(t) = 0$ directly, without using part a)

Extra Credit. [1 point] Show that

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{r}(t)\|^2\right) = \mathbf{r}(t)\cdot\mathbf{r}'(t)$$

$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

$$\frac{d}{dt}\left(\frac{1}{2}\|\vec{r}(t)\|^2\right) = \frac{d}{dt}\left(\frac{1}{2}\vec{r}(t)\cdot\vec{r}(t)\right)$$

$$= \frac{1}{2}\left(\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)\right)$$

product rule

$$= \frac{1}{2}\cdot 2\,\vec{r}(t)\cdot\vec{r}\,'(t) = \vec{r}(t)\cdot\vec{r}\,'(t)$$

EXTRA SPACE FOR ANSWERS