Name: SOLUTIONS

Math 253 Calculus III (Bueler) Thursday, 6 April 2023

Midterm Exam 2

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

1. *(10 pts)* Find an equation of the tangent plane of the surface $z = \ln(10x^2 + 2y^2 + 1)$ at the point $P(0, 0, 0)$.

$f(x,y) = \ln(10x^2+2y^2+1)$

$z_0 = f(0,0) = \ln(0+0+1) = 0$ ✓

$f_x = (10x^2+2y^2+1)^{-1}(20x) \Rightarrow f_x(0,0) = 0$

$f_y = (10x^2+2y^2+1)^{-1}(4y) \Rightarrow f_y(0,0) = 0$

$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= 0 + 0 + 0 \quad = 0$

2. **(a)** *(5 pts)* Suppose $f(u, v)$ is a function of two variables, and that, in turn, $u = u(r, s)$ and $v = v(r, s)$. Write down the chain rule which computes $\frac{\partial f}{\partial s}$:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s}$$

(b) *(5 pts)* Specifically suppose $f(u, v) = uv$, $u(r, s) = r \cos s$, and $v(r, s) = r \sin s$. Compute the following partial derivative, and express your answer as a simplified expression in variables r, s.

$$\frac{\partial f}{\partial s} = v \cdot (r(-\sin s)) + u \cdot (r\cos s)$$

$= -r\sin s \cdot r\sin s + r\cos s \cdot r\cos s$

$= r^2(\cos^2 s - \sin^2 s)$

$= r^2 \cos(2s)$

3. Consider the function $f(x, y) = x^3 + y^3 - 12x - 15y + 7$.

(a) (*5 pts*) Compute the gradient:

$\nabla f(x, y) = \langle 3x^2 - 12, 3y^2 - 15 \rangle$

(b) (*5 pts*) Find all of the critical points. Write each one as a pair (x, y).

$$\begin{cases} 3x^2 - 12 = 0 \\ 3y^2 - 15 = 0 \end{cases}$$

$\Leftrightarrow$

$x = \pm 2$

$y = \pm\sqrt{5}$

critical points:

$(+2, +\sqrt{5})$

$(+2, -\sqrt{5})$

$(-2, +\sqrt{5})$

$(-2, -\sqrt{5})$

(c) (*5 pts*) Use the second derivative test to classify all of the critical points, as local maximum, local minimum, or saddle point.

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = f_{yx} = 0$

$\Downarrow$

$D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$

$(+2, +\sqrt{5})$: $D > 0$, $f_{xx} > 0$ $\therefore$ local min.

$(+2, -\sqrt{5})$: $D < 0$ $\therefore$ saddle

$(-2, +\sqrt{5})$: $D < 0$ $\therefore$ saddle

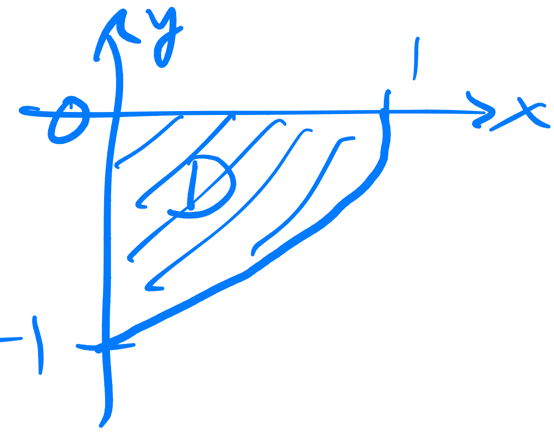
$(-2, -\sqrt{5})$: $D > 0$, $f_{xx} < 0$ $\therefore$ local max

(d) (*3 pts*) Consider the square $D = [-10, 10] \times [-10, 10]$. Does the absolute maximum of $f(x, y)$ over D occur at one of the critical points found in part **(b)**? State the answer **yes** or **no**, and explain in one sentence.

(*Hint.* You do not need to find the absolute maximum! But consider values of $f(x, y)$ when answering.)

No. The value of $f(10,10) = 10^3 + 10^3 - 120 - 150 + 7 > 1000$ is much bigger than the f-values at the critical points.

4. **(a)** (*5 pts*) Sketch the region
$D = \left\{(x, y) \,\middle|\, -1 \le y \le 0,\ 0 \le x \le 1 - y^2\right\}$.



(b) (*10 pts*) Compute and simplify the double integral over the region D in part **(a)**:

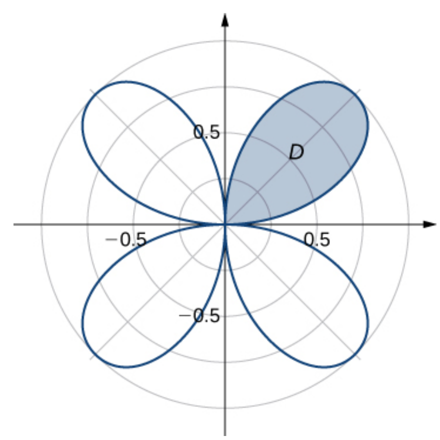
$$\iint_D xy\,dA = \int_{y=-1}^{y=0}\int_{x=0}^{x=1-y^2} xy\,dx\,dy$$

$$= \int_{-1}^{0} y\left[\frac{x^2}{2}\right]_0^{1-y^2} dy = \frac{1}{2}\int_{-1}^{0} y(1-y^2)^2\,dy$$

$$= \frac{1}{2}\int_{-1}^{0} y - 2y^3 + y^5\,dy = \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{2} + \frac{y^6}{6}\right]_{-1}^{0}$$

$$= \frac{1}{2}\left(0 - \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \tfrac{1}{6}\right)\right) = -1/12$$

5. (*10 pts*) Set up, and then evaluate, a double integral for the area of one leaf of the rose $r = \sin(2\theta)$, as shown in the figure.



$$A_D = \iint_D 1\,dA$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{r=\sin(2\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\sin(2\theta)} d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 - \cos 4\theta\,d\theta = \frac{1}{4}\left[\theta - \frac{\overset{0}{\cancel{\sin(4\theta)}}}{4}\right]_0^{\pi/2} = \frac{\pi}{8}$$

6. (*10 pts*) Define $B = \left\{(x,y,z) \,\middle|\, 0 \leq x \leq 1,\ 0 \leq y \leq \pi,\ -1 \leq z \leq 1\right\}$, a rectangular solid box. Evaluate the triple integral:

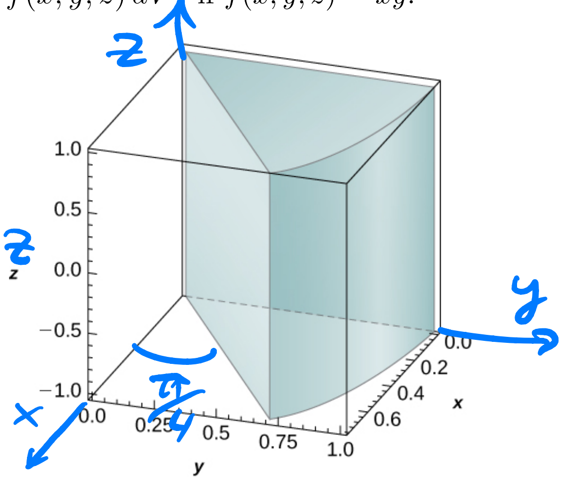
$$\iiint_B x \sin y \, dV = \int_{x=0}^{1} \int_{y=0}^{\pi} \int_{z=-1}^{1} x \sin y \, dz\, dy\, dx$$

$$= \left(\int_0^1 x\,dx\right)\left(\int_0^{\pi} \sin y\,dy\right)\left(\int_{-1}^{1} dz\right)$$

$$= \frac{1}{2} \cdot \Big[-\cos y\Big]_0^{\pi} \cdot 2 = -(-1)-(-1) = 2$$

$x \leq y$

7. (*10 pts*) The region $E = \left\{(x,y,z) \,\middle|\, x^2+y^2 \leq 1,\ x \geq 0,\ x \cancel{\geq} y,\ -1 \leq z \leq 1\right\}$ is shown below. Use **cylindrical coordinates** to evaluate the triple integral $\iiint_E f(x,y,z)\,dV$, if $f(x,y,z) = xy$.



$$\iiint_E xy\,dV$$

$$= \int_{z=-1}^{1} \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{1} r\cos\theta\, r\sin\theta \cdot r\,dr\,d\theta\,dz$$

$$= \left(\int_{-1}^{1} dz\right)\left(\int_{\pi/4}^{\pi/2} \cos\theta\sin\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

$$= 2 \cdot \int_{1/\sqrt{2}}^{1} u\,du \cdot \frac{1}{4}$$

$$= \frac{1}{2}\left[\frac{u^2}{2}\right]_{1/\sqrt{2}}^{1} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$$

8. (*10 pts*) Find the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ over the unit sphere $S = \left\{(x,y,z)\,\middle|\, x^2+y^2+z^2 \le 1\right\}$.

(*Hints.* What coordinates would make this easiest? Yes, you *may* use the fact that the volume of the unit sphere is $4\pi/3$; there is no need to justify it.)

spherical!

$$av_f = \frac{1}{V_S} \iiint_S x^2+y^2+z^2 \, dV$$

$$= \frac{3}{4\pi} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{\rho=0}^{1} \rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{3}{4\pi} \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\pi} \sin\varphi \, d\varphi\right) \left(\int_0^1 \rho^4 \, d\rho\right)$$

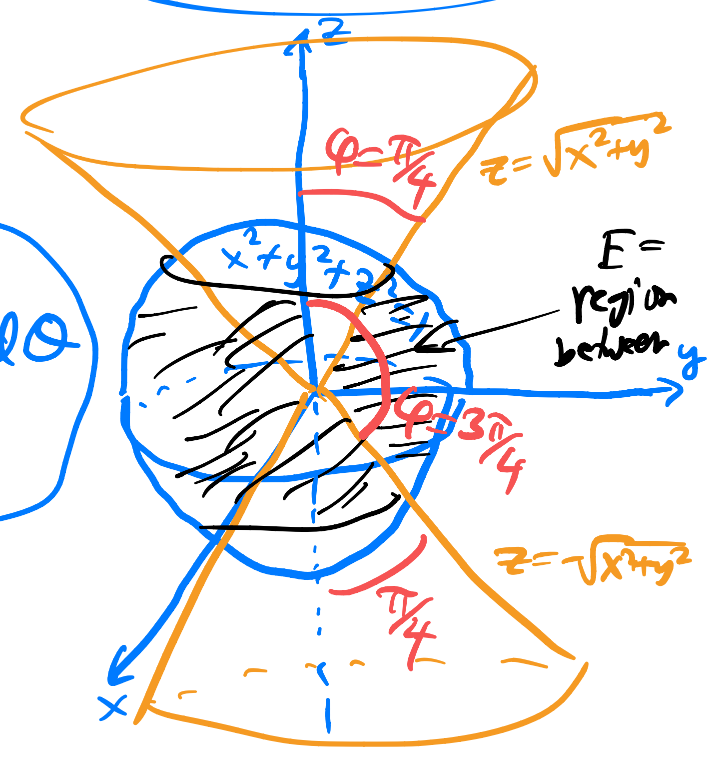
$$= \frac{3}{\cancel{4\pi}_{\,2}} \cdot \cancel{2\pi} \cdot \left[-\cos\varphi\right]_0^{\pi} \left[\frac{\rho^5}{5}\right]_0^1 = \frac{3}{2} \cdot 2 \cdot \frac{1}{5} = \frac{3}{5}$$

9. (*7 pts*) Use **spherical coordinates** to fully set up a triple integral ~~for the volume which~~ is outside the cone $z^2 = x^2 + y^2$ but inside the unit sphere $x^2+y^2+z^2 = 1$. Do not evaluate the integral.

(*Hint.* Start by drawing a decent sketch!)

$$V = \iiint_E 1 \, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=\pi/4}^{3\pi/4} \int_{\rho=0}^{1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



Extra Credit. (*3 pts*) The equation $x^2 + y^2 = 9$ is a cylinder. Convert this equation to **spherical coordinates**, and write your simplified answer in the form $\rho = f(\varphi, \theta)$.

in spherical: $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$

$$x^2 + y^2 = 9 \iff \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta = 9$$

$$\rho^2 \sin^2\varphi (\cos^2\theta + \sin^2\theta) = 3^2$$

$$\rho^2 \sin^2\varphi = 3^2 \leftarrow$$

$\rho \geq 0$ and $\sin\varphi \geq 0$ apply to all of 3D)

$$\rho \sin\varphi = 3$$

$$\rho = \frac{3}{\sin\varphi} = 3\csc\varphi$$

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