Name: SOLUTIONS

Math 253 Calculus III (Bueler) Thursday, 23 February 2023

Midterm Exam 1

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

1. Suppose we have three vectors, $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Compute the following quantities which are either scalars or vectors. You can write the vectors using either component notation or standard unit vector notation.

(a) (*5 pts*) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} =$

$$\vec{a} \cdot \vec{b} = \langle 1,-1,0\rangle \cdot \langle 0,1,3\rangle = -1$$

$$(\vec{a} \cdot \vec{b})\vec{c} = -\langle -1,2,-4\rangle = \langle 1,-2,4\rangle$$

(b) (*5 pts*) a unit vector in the direction of $\mathbf{c}$: $\mathbf{u} = \left\langle \frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}} \right\rangle$

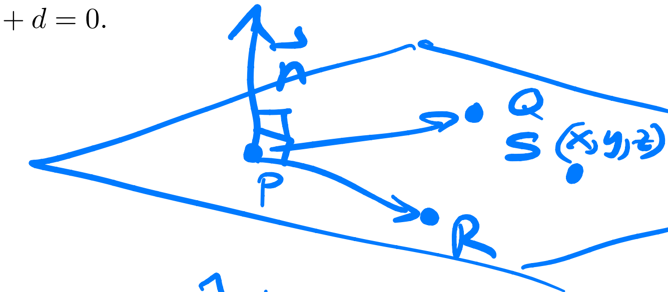
$$\|\vec{c}\| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$$

(c) (*5 pts*) the angle between vectors $\mathbf{a}$ and $\mathbf{b}$: $\theta =$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{-1}{\sqrt{2}\sqrt{1^2+3^2}} = \frac{-1}{\sqrt{2}\sqrt{2}\sqrt{5}}$$

$$\theta = \arccos\left(\frac{-1}{2\sqrt{5}}\right)$$

2. **(a)** (*10 pts*) Find a general equation of the plane through the three points $P(3, -1, 2)$, $Q(1, 0, 1)$, and $R(0, -1, 1)$. Express your answer in the form $ax + by + cz + d = 0$.



$$\vec{PQ} = \langle -2, 1, -1 \rangle$$

$$\vec{PR} = \langle -3, 0, -1 \rangle$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & 0 & -1 \end{vmatrix}$$

$$= (-1)\hat{i} - (2-3)\hat{j} + (+3)\hat{k} = \langle -1, +1, +3 \rangle$$

plane is $\quad \vec{n} \cdot \vec{PS} = 0$

$$\langle -1, 1, 3 \rangle \cdot \langle x-3, y+1, z-2 \rangle = 0$$

$$-(x-3) + y + 1 + 3z - 6 = 0$$

$$-x + y + 3z - 2 = 0$$

(b) (*5 pts*) Consider the same three points as in part **(a)**. Find the area of the triangle PQR.

$$A_\Delta = \frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$$

$$= \frac{1}{2} \|\vec{n}\| = \frac{1}{2}\sqrt{1^2 + 1^2 + 3^2}$$

$$= \frac{\sqrt{11}}{2}$$

3. Suppose $z = \ln(xy + y^4)$. Compute the following partial derivatives. There is no need to simplify your answers.

(a) (*5 pts*) $\dfrac{\partial z}{\partial y} = \dfrac{1}{xy+y^4} \cdot (x + 4y^3)$

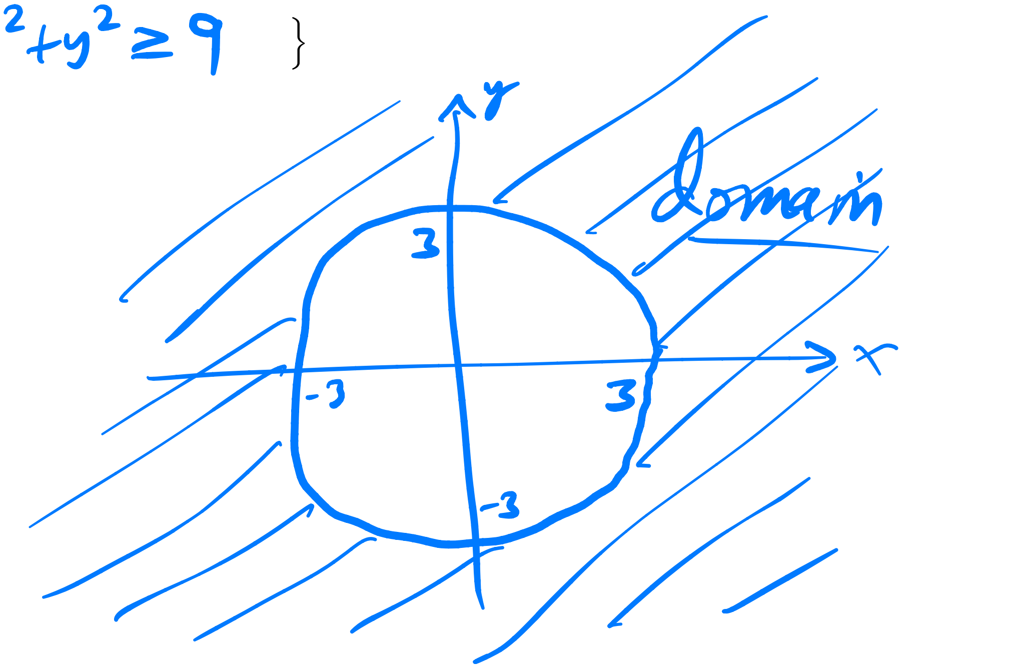
$$= \frac{x+4y^3}{xy+y^4}$$

(b) (*5 pts*) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1\cdot(xy+y^4) - (x+4y^3)\,y}{(xy+y^4)^2}$

$$= \frac{\cancel{xy}+y^4-\cancel{xy}-4y^4}{(xy+y^4)^2} = \frac{-3y^4}{(xy+y^4)^2}$$

4. (*5 pts*) Find and sketch (shade in) the domain of the function $f(x, y) = \sqrt{x^2 + y^2 - 9}$. Fill in the set notation below.

(domain of f) $= \{(x, y) \mid\ x^2+y^2 \geq 9 \ \}$



5. (*5 pts*) Find a vector equation of the line passing through the two points $P(4, 0, 5)$ and $Q(2, 3, 1)$.

$\vec{r}_0 = \langle 4, 0, 5 \rangle$

$\vec{v} = \overrightarrow{PQ} = \langle -2, 3, -4 \rangle$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$= \langle 4-2t, 3t, 5-4t \rangle$

6. (*10 pts*) Suppose that a moving particle has position function $\mathbf{r}(t) = \langle e^{-t}, t, t^2 \rangle$. Calculate the tangent line to the curve $\mathbf{r}(t)$ at $t = 1$. (*Hint.* The answer can be vector-valued or parametric.)

$\vec{r}\,'(t) = \langle -e^{-t}, 1, 2t \rangle$

$\vec{r}_0 = \vec{r}(1) = \langle e^{-1}, 1, 1 \rangle$

$\vec{v} = \vec{r}\,'(1) = \langle -e^{-1}, 1, 2 \rangle$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$= \langle \frac{1}{e} - \frac{1}{e}t, 1+t, 1+2t \rangle$

$= \langle \frac{1}{e}(1-t), 1+t, 1+2t \rangle$

7. (*5 pts*) Compute the arc length of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t = -2$ to $t = 0$.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$s = \int_{-2}^{0} \|\vec{r}\,'(t)\| dt = \int_{-2}^{0} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \int_{-2}^{0} \sqrt{2}\, dt = \sqrt{2} \int_{-2}^{0} dt = \sqrt{2} \cdot 2$$

$$= 2\sqrt{2}$$

8. (*5 pts*) Write the curve (graph) $y = f(x)$ as a vector-valued curve $\mathbf{r}(t)$.

$$x(t) = t$$
$$y(t) = f(t)$$

so $\vec{r}(t) = \langle t, f(t) \rangle$

9. (*5 pts*) Compute the limit:

$$\lim_{(x,y)\to(1,1)} \frac{xy - x}{y^2 - 1} = \lim_{(x,y)\to(1,1)} \frac{x\cancel{(y-1)}}{\cancel{(y-1)}(y+1)}$$

"$\frac{0}{0}$"

$$= \lim_{(x,y)\to(1,1)} \frac{x}{y+1} = \frac{1}{1+1} = \frac{1}{2}$$

10. **(a)** (*5 pts*) Find $\mathbf{T}(t)$, the unit tangent vector, for the circle $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$.

$$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t\rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle -3\sin t, 3\cos t\rangle}{\sqrt{3^2\sin^2 t + 3^2\cos^2 t}}$$

$$= \frac{\langle -3\sin t, 3\cos t\rangle}{3} = \langle -\sin t, \cos t\rangle$$

(b) (*5 pts*) Compute the curvature of the curve $\mathbf{r}(t)$ in part **(a)**, at the point $t = 0$.

the circle has radius $R=3$ so curvature is

$$\kappa = \frac{1}{R} = \frac{1}{3}$$

[or: $\kappa(t) = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dots = \dfrac{1}{3}$]

Extra Credit. (*3 pts*) Show that the following limit does not exist:

$$\circledast \quad \lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$$

along $y=0$: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{3x^2+0} = 0$

along $y=x$: $\displaystyle\lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2+x^2} = \frac{1}{2}$

but $\circledast$ is an "any direction" limit, so it does not exist

11. (*5 pts*) Are the two planes $\overbrace{x - 2y + 3z = 5}^{A}$ and $\overbrace{-2x + 4y - 6z = 0}^{B}$ parallel? If so, explain why. If not, find the angle between the planes.

$$\left.\begin{aligned} \vec{n}_A &= \langle 1, -2, 3 \rangle \\ \vec{n}_B &= \langle -2, 4, -6 \rangle \end{aligned}\right\} \text{ so } \vec{n}_B = -2\,\vec{n}_A,$$

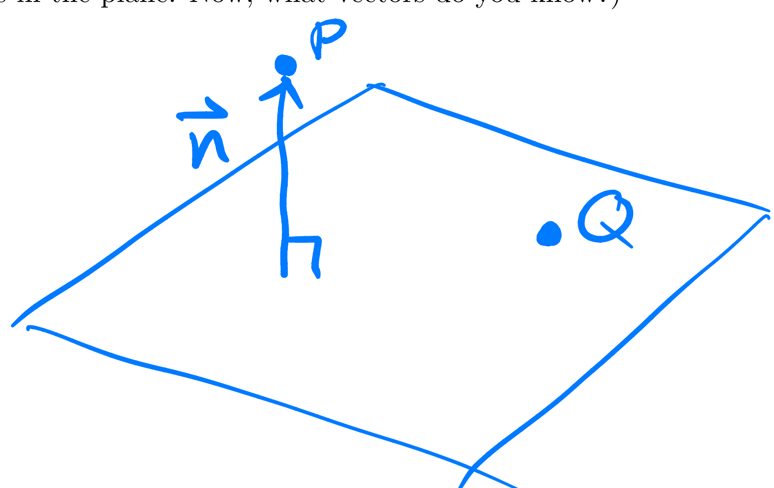
thus the planes are parallel

[or compute $\vec{n}_A \times \vec{n}_B = \vec{0}$

or compute $\cos\theta = \dfrac{\vec{n}_A \cdot \vec{n}_B}{\|\vec{n}_A\|\,\|\vec{n}_B\|} = -1$]

12. (*10 pts*) Find the distance from the point $P(0, 0, 1)$ to the plane $x + 2y + 3z = 4$. (*Hint.* To start, draw a sketch and find a concrete point which is in the plane. Now, what vectors do you know?)

$Q(4,0,0)$ is in plane



$\vec{n} = \langle 1, 2, 3 \rangle$

$\overrightarrow{QP} = \langle -4, 0, 1 \rangle$

$$d = \left|\operatorname{comp}_{\vec{n}} \overrightarrow{QP}\right| = \left|\frac{\overrightarrow{QP} \cdot \vec{n}}{\|\vec{n}\|}\right| = \left|\frac{-4+0+3}{\sqrt{1^2+2^2+3^2}}\right|$$

$$= \frac{1}{\sqrt{14}}$$

You may find the following curvature formulas useful. However, there are many other formulas, not listed here, which you should have in memory.

$$\begin{aligned} \kappa(s) &= \left\| \frac{d\mathbf{T}}{ds} \right\| && \\ \kappa(t) &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} && \text{for curves } \mathbf{r}(t) \\ \kappa(x) &= \frac{|f''(x)|}{\left(1+f'(x)^2\right)^{3/2}} && \text{for curves } y = f(x) \end{aligned}$$

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