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Name: SOLUTIONS

Math 253 Calculus III (Bueler) Wednesday, 3 May 2023

Final Exam

No book, electronics, calculator, or internet access. $\frac{1}{2}$ sheet of notes allowed; double-sided okay! 125 points possible. 120 minutes maximum.

1. (*8 pts*) Match the vector fields $\mathbf{F}$ with the plots. Write labels **(a)** through **(d)** in the spaces.

(c) $\mathbf{F}(x,y) = \langle x, -y \rangle$

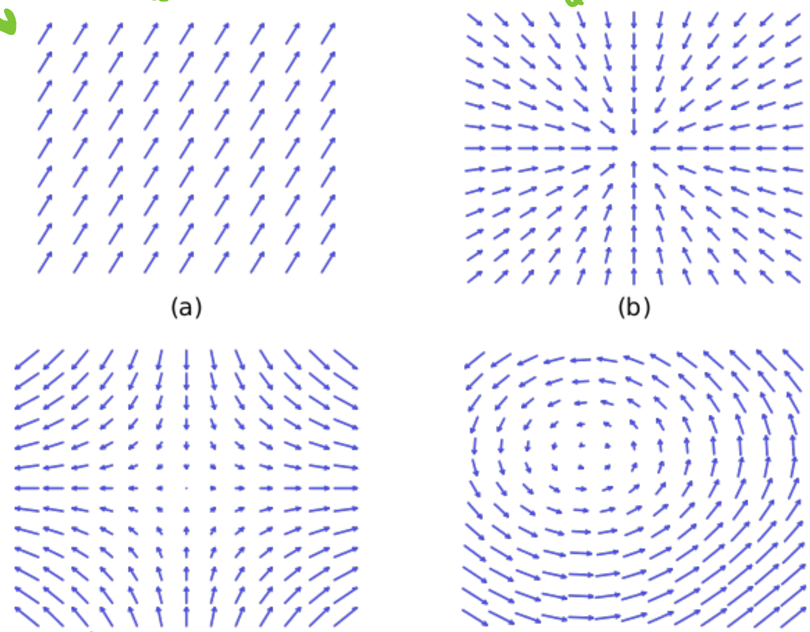
(b) $\mathbf{F}(x,y) = \left\langle \frac{-3x}{\sqrt{x^2+y^2}}, \frac{-3y}{\sqrt{x^2+y^2}} \right\rangle$

(a) $\mathbf{F}(x,y) = \langle 2, 4 \rangle$

(d) $\mathbf{F}(x,y) = \langle 4\cos(\frac{y}{3}+\frac{\pi}{4}), 4\sin(\frac{y}{3}+\frac{\pi}{4}) \rangle$

constant length and direction

constant length, radially inward



(a) (b) (c) (d)

y-component flipped versus $\langle x, y \rangle$

2. (*10 pts*) Determine whether $\mathbf{F}$ is a conservative vector field. If it is, find f so that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j} = \langle P, Q \rangle$$

$P_y = e^x + \cos y$

$Q_x = e^x + \cos y$ ✓

yes, conservative

$f_x = P = ye^x + \sin y$

$f(x,y) = ye^x + x\sin y + g(y)$

$e^x + x\cos y = Q = f_y = e^x + x\cos y + g'(y) \iff g'(y) = 0$

$\iff g(y) = c$

$f(x,y) = ye^x + x\sin y$ "+c" optional

$P = x^2 + y, \quad Q = 3x - y^2$

3. (*10 pts*) Suppose $\mathbf{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for *any* positively-oriented, closed, simple curve C enclosing a region D which has area 6. (*Hint.* Green's Theorem)

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy \overset{GT}{=} \iint_D Q_x - P_y \, dA$$

$$= \iint_D 3 - 1 \, dA = 2 A_D = 12$$

4. (*10 pts*) Evaluate the line integral if C is the line segment from $(0,0)$ to $(5,4)$:

$$\int_C xe^y \, ds = \int_0^1 x(t) e^{y(t)} \frac{ds}{dt} \, dt$$

$\vec{r}(t) = t\langle 5, 4 \rangle$

$0 \leq t \leq 1$

$\vec{r}\,'(t) = \langle 5, 4 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{25 + 16} = \sqrt{41}$

$$= \int_0^1 5t \, e^{4t} \sqrt{41} \, dt$$

$$= 5\sqrt{41} \int_0^1 t e^{4t} \, dt$$

integrate by parts:

$u = t \quad v = \frac{1}{4} e^{4t}$

$du = dt \quad dv = e^{4t} dt$

$$= 5\sqrt{41} \left(\left[\frac{1}{4} t e^{4t}\right]_0^1 - \int_0^1 \frac{1}{4} e^{4t} dt \right)$$

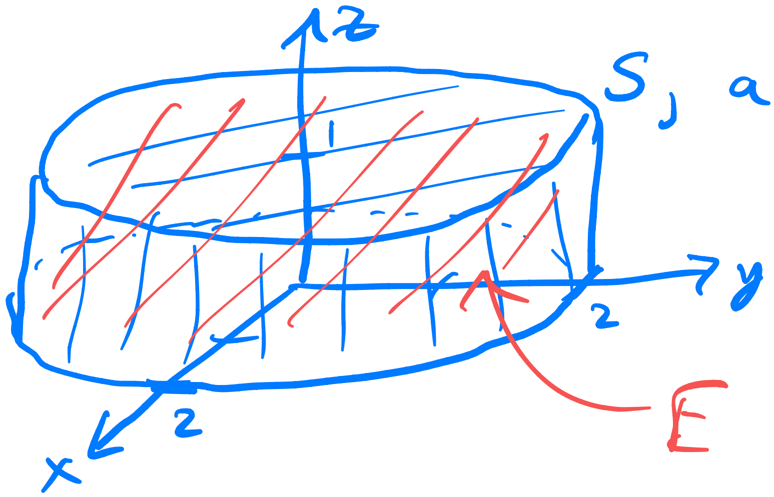
$$= 5\sqrt{41} \left(\frac{1}{4} e^4 - \frac{1}{16} \left[e^{4t} \right]_0^1 \right) = 5\sqrt{41} \left(\frac{1}{4} e^4 - \frac{1}{16} e^4 + \frac{1}{16} \right)$$

$$= 5\sqrt{41} \left(\frac{3}{16} e^4 + \frac{1}{16} \right)$$

5. Consider the closed surface S which is formed by the equations $x^2 + y^2 = 4$, $z = 0$, and $z = 1$.

(a) (*4 pts*) Sketch the surface S. (*Remember to label the axes and indicate scale on each axis. Sketch reasonably large!*)

$x^2+y^2=2^2$ is cylinder



S, a closed surface

E is enclosed by S

(b) (*6 pts*) Now suppose $\mathbf{F} = \langle x, y, z^2 - 1\rangle$ is a vector field. Use the **divergence theorem** to compute

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV = \iiint_E 1+1+2z\,dV$$

$$= 2\int_0^{2\pi}\int_0^2\int_0^1 1+z\; r\,dz\,dr\,d\theta = 4\pi\left(\int_0^2 r\,dr\right)\left(\int_0^1 1+z\,dz\right)$$

$$= 4\pi\cdot\frac{2^2}{2}\cdot\left[z+\frac{z^2}{2}\right]_0^1 = 8\pi\left(1+\frac{1}{2}\right) = 8\pi\cdot\frac{3}{2}$$

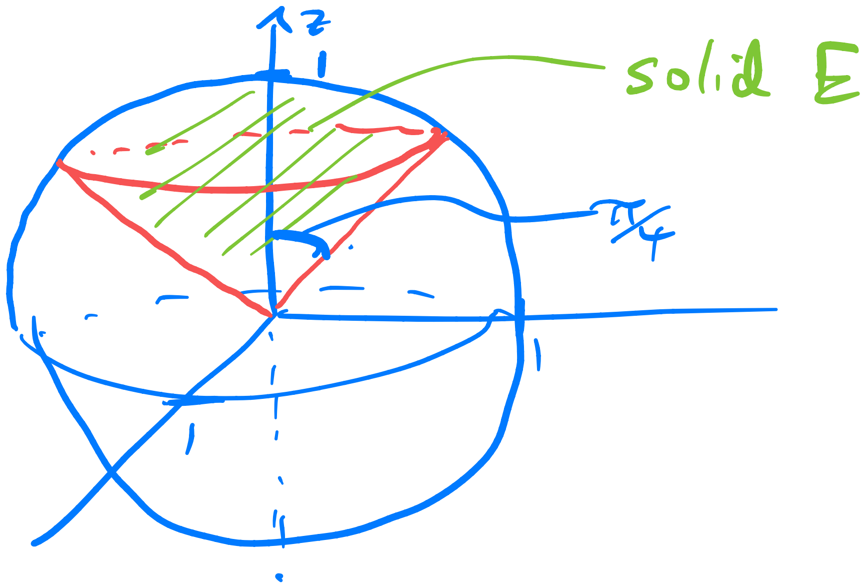
$$= 12\pi$$

6. (*8 pts*) Find a unit vector that is orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$.

$$\vec{v} = \langle 1,1,0\rangle \times \langle 1,0,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 1,-1,-1\rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,-1,-1\rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right\rangle$$

7. **(a)** (*4 pts*) Sketch the solid which is above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=1$. (*Label the axes and indicate scale on each axis. Sketch reasonably large!*)



(b) (*6 pts*) Use **spherical coordinates** to find the volume of the solid in part **(a)**.

$$V = \iiint_E 1\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \overbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}^{dV \text{ in spherical}}$$

$$= 2\pi \left(\int_0^1 \rho^2 \, d\rho\right)\left(\int_0^{\pi/4} \sin\phi \, d\phi\right)$$

$$= 2\pi \cdot \frac{1}{3} \cdot \Big[-\cos\phi\Big]_0^{\pi/4} = \frac{2\pi}{3}\left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

8. (*10 pts*) Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$. Simplify to the form $ax + by + cz + d = 0$.

$$\vec{n} = \langle 5, -1, -1 \rangle$$

plane: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle 5, -1, -1 \rangle \cdot \langle x-1, y+1, z+1 \rangle = 0$

$$\iff 5(x-1) - (y+1) - (z+1) = 0$$

$$\iff 5x - y - z - 7 = 0$$

9. Suppose $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + (z^2 + xy)\mathbf{k}$.

(a) (*5 pts*) Compute and simplify the divergence $\nabla \cdot \mathbf{F}$.

$$\nabla \cdot \vec{F} = 1 + 2y + 2z$$

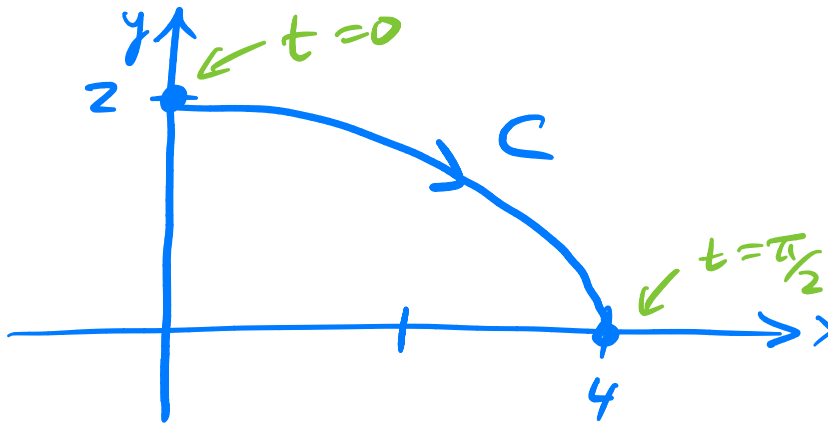
(b) (*5 pts*) Compute and simplify the curl $\nabla \times \mathbf{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y^2 & z^2+xy \end{vmatrix} = (x-0)\hat{\imath} - (y-0)\hat{\jmath} + (0-0)\hat{k}$$

$$= \langle x, -y, 0 \rangle$$

10. **(a)** (*5 pts*) Sketch the plane curve C with the given vector equation, indicating its orientation. (*Label the axes and indicate scale on each axis. Sketch reasonably large!*)

$$\mathbf{r}(t) = 4\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}, \qquad 0 \le t \le \frac{\pi}{2}$$



(b) (*5 pts*) Compute the unit tangent vector field $\mathbf{T}(t)$ for the curve C in part **(a)**.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 4\cos t, -2\sin t\rangle}{\sqrt{4^2\cos^2 t + 2^2\sin^2 t}}$$

$$= \frac{1}{\sqrt{3\cos^2 t + 1}}\langle 2\cos t, -\sin t\rangle$$

either is fine

11. (*9 pts*) Compute and simplify the linearization $L(x, y)$ of the function at the point:

$$f(x, y) = \sqrt{xy}, \qquad (1, 4)$$

$$f = (xy)^{1/2} \Rightarrow f_x = \frac{1}{2}(xy)^{-1/2}\cdot y = \frac{\sqrt{y}}{2\sqrt{x}}$$

$$f_y = \frac{1}{2}(xy)^{-1/2}\cdot x = \frac{\sqrt{x}}{2\sqrt{y}}$$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= 2 + \frac{\sqrt{4}}{2\cdot 1}(x-1) + \frac{1}{2\cdot 2}(y-4) = 2 + x - 1 + \frac{1}{4}(y-4)$$

$$= x + \frac{y}{4}$$

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12. Suppose

$$w = xy + yz + zx, \qquad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$$

(a) (*5 pts*) State the chain rule for $\dfrac{\partial w}{\partial \theta}$ which applies in this case.

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

(b) (*5 pts*) Compute $\dfrac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$. Simplify the answer to a number.

$x = 0$
$y = 2$
$z = \pi$

$$\frac{\partial w}{\partial x} = y + z, \quad \frac{\partial w}{\partial y} = x + z, \quad \frac{\partial w}{\partial z} = y + x$$

$$\therefore \frac{\partial w}{\partial \theta} = (y+z)\cdot(-r\sin\theta) + (x+z)\cdot(r\cos\theta) + (y+x)\cdot r$$

$$= (2+\pi)\cdot(-2\cdot 1) + \overset{0}{\cancel{(0+\pi)(2\cdot 0)}} + (0+2)\cdot 2$$

$$= -4 - 2\pi + 4 = -2\pi$$

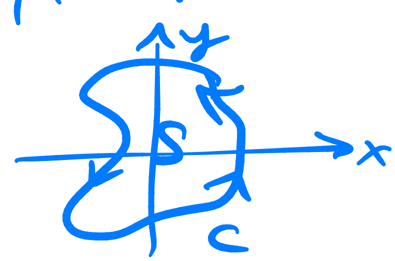
Extra Credit I. (*2 pts*) Stokes' theorem says that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$ for any surface S in 3D with oriented boundary C. Explain, in sentences and equations, using correct notation, the situation in which this theorem becomes Green's theorem. (*A sketch may help, too.*)

Suppose S is a surface in the x,y plane.

Then $\vec{N} = \hat{k}$ is a normal direction,

and $\nabla \times \vec{F} = \langle Q_z - R_y, R_x - P_z, Q_x - P_y \rangle$,

so $(\nabla \times \vec{F}) \cdot \hat{k} = Q_x - P_y$. So



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{N}\, dS = \iint_S Q_x - P_y\, dA = \oint_C \vec{F} \cdot d\vec{r}_s$$

Green's theorem!

13. **(a)** (*5 pts*) Find all critical points of the function:

$f(x,y) = 2 - x^4 + 2x^2 - y^2$

$\nabla f = \langle -4x^3 + 4x, -2y \rangle \Rightarrow$

$-4x^3 + 4x = 0$
$-2y = 0$

$\Longleftrightarrow x(-x^2+1) = 0$
$y = 0$

$\Longleftrightarrow x=0$ or $x = \pm 1$
and $y = 0$

$(0,0)$
$(+1,0)$
$(-1,0)$

(b) (*5 pts*) Find all local maxima, local minima, and saddle points of the function in part **(a)**.

$D = f_{xx} f_{yy} - f_{xy}^2 = (-12x^2+4)(-2) - 0$
$= 8(3x^2-1)$

crit. pt.	D	f_{xx}	type
$(0,0)$	$-$		saddle
$(+1,0)$	$+$	$-$	local max
$(-1,0)$	$+$	$-$	local max

Extra Credit II. (*2 pts*) Assume $\mathbf{F}(x,y)$ is a conservative vector field defined on a open, connected region D. Fix any point (a,b) in D. Explain why the formula $f(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$ ⊗ defines a *function* on D.

Since $\vec{F}$ is conservative, $\vec{F} = \nabla f$ for some $f(x,y)$. Therefor

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(\text{end } C) - f(\text{start } C)$$

is path-independent. So ⊗ makes sense: it defines one value $f(x,y)$. [And $\nabla f = \vec{F}$ in fact.]