Name: ______________________________

Math 253 Calculus III (Bueler) Wednesday, 3 May 2023

Final Exam

No book, electronics, calculator, or internet access. $\frac{1}{2}$ sheet of notes allowed; double-sided okay! 125 points possible. 120 minutes maximum.

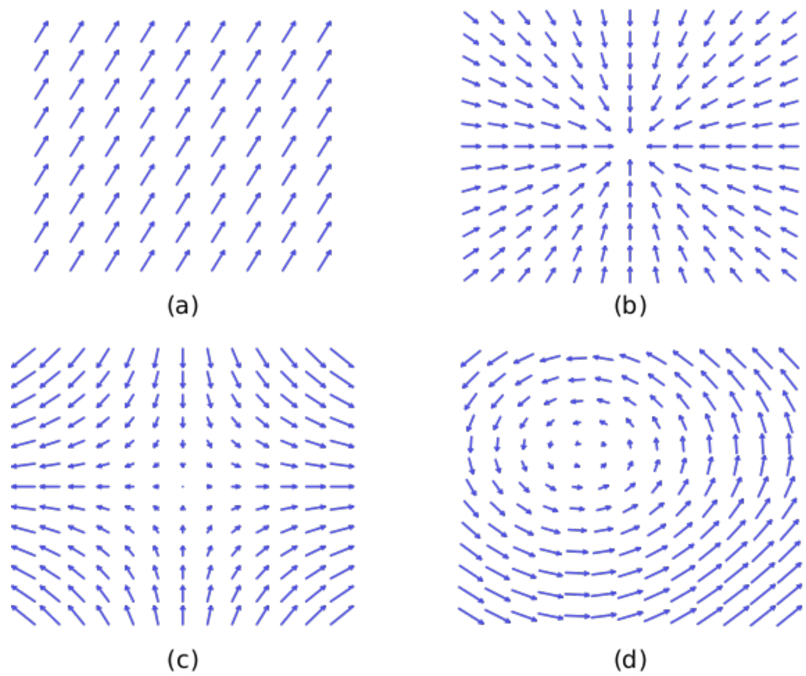
1. (*8 pts*) Match the vector fields $\mathbf{F}$ with the plots. Write labels **(a)** through **(d)** in the spaces.

______ $\mathbf{F}(x,y) = \langle x, -y \rangle$

______ $\mathbf{F}(x,y) = \left\langle \frac{-3x}{\sqrt{x^2+y^2}}, \frac{-3y}{\sqrt{x^2+y^2}} \right\rangle$

______ $\mathbf{F}(x,y) = \langle 2, 4 \rangle$

______ $\mathbf{F}(x,y) = \left\langle 4\cos(\frac{y}{3} + \frac{\pi}{4}), 4\sin(\frac{y}{3} + \frac{\pi}{4}) \right\rangle$



2. (*10 pts*) Determine whether $\mathbf{F}$ is a conservative vector field. If it is, find f so that $\mathbf{F} = \nabla f$.

$\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$

3. (*10 pts*) Suppose $\mathbf{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for *any* positively-oriented, closed, simple curve C enclosing a region D which has area 6. (*Hint.* Green's Theorem)

4. (*10 pts*) Evaluate the line integral if C is the line segment from $(0,0)$ to $(5,4)$:

$$\int_C xe^y \, ds =$$

5. Consider the closed surface S which is formed by the equations $x^2 + y^2 = 4$, $z = 0$, and $z = 1$.

(a) (*4 pts*) Sketch the surface S. (*Remember to label the axes and indicate scale on each axis. Sketch reasonably large!*)

(b) (*6 pts*) Now suppose $\mathbf{F} = \langle x, y, z^2 - 1 \rangle$ is a vector field. Use the **divergence theorem** to compute

$$\oiint_S \mathbf{F} \cdot \mathbf{N}\, dS =$$

6. (*8 pts*) Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

7. **(a)** (*4 pts*) Sketch the solid which is above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 1$. (*Label the axes and indicate scale on each axis. Sketch reasonably large!*)

(b) (*6 pts*) Use **spherical coordinates** to find the volume of the solid in part **(a)**.

8. (*10 pts*) Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$. Simplify to the form $ax + by + cz + d = 0$.

9. Suppose $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + (z^2 + xy)\mathbf{k}$.

(a) (*5 pts*) Compute and simplify the divergence $\nabla \cdot \mathbf{F}$.

(b) (*5 pts*) Compute and simplify the curl $\nabla \times \mathbf{F}$.

10. **(a)** (*5 pts*) Sketch the plane curve C with the given vector equation, indicating its orientation. (*Label the axes and indicate scale on each axis. Sketch reasonably large!*)

$$\mathbf{r}(t) = 4\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}, \qquad 0 \le t \le \frac{\pi}{2}$$

(b) (*5 pts*) Compute the unit tangent vector field $\mathbf{T}(t)$ for the curve C in part **(a)**.

11. (*9 pts*) Compute and simplify the linearization $L(x, y)$ of the function at the point:

$$f(x, y) = \sqrt{xy}, \qquad (1, 4)$$

12. Suppose

$$w = xy + yz + zx, \qquad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$$

(a) (*5 pts*) State the chain rule for $\dfrac{\partial w}{\partial \theta}$ which applies in this case.

(b) (*5 pts*) Compute $\dfrac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$. Simplify the answer to a number.

Extra Credit I. (*2 pts*) Stokes' theorem says that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$ for any surface S in 3D with oriented boundary C. Explain, in sentences and equations, using correct notation, the situation in which this theorem becomes Green's theorem. (*A sketch may help, too.*)

13. **(a)** (*5 pts*) Find all critical points of the function:

$f(x, y) = 2 - x^4 + 2x^2 - y^2$

(b) (*5 pts*) Find all local maxima, local minima, and saddle points of the function in part **(a)**.

Extra Credit II. (*2 pts*) Assume $\mathbf{F}(x, y)$ is a conservative vector field defined on a open, connected region D. Fix any point (a, b) in D. Explain why the formula $f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$ defines a *function* on D.