SOLUTIONS

Worksheet: Improper integrals

CORRECTED

Compute these integrals with friends! Also, please carefully write the limit, for example

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

CORRECTED

converges

A. $\int_2^\infty \frac{1}{9+x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{3^2+x^2}\,dx = \lim_{t\to\infty} \int_{\arctan(2/3)}^{\arctan(t/3)} \frac{3\sec^2\theta\,d\theta}{3^2\sec^2\theta}$

$3\tan\theta = x$

$$= \frac{1}{3}\lim_{t\to\infty}\int_{\mathrm{atan}(2/3)}^{\mathrm{atan}(t/3)} d\theta = \frac{1}{3}\lim_{t\to\infty} \mathrm{atan}(t/3) - \mathrm{atan}(2/3)$$

$$= \frac{1}{3}(\pi/2 - \mathrm{atan}(2/3))$$

converges

B. $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\int_t^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0$

$$= \lim_{t\to-\infty} e^0 - e^t = \lim_{t\to-\infty} 1 - e^t = 1 - 0 = 1$$

converges

C. $\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx = \lim_{t\to0^+}\int_t^1 x^{-1/4}\,dx = \lim_{t\to0^+}\left[\frac{4}{3}x^{3/4}\right]_t^1$

$$= \lim_{t\to0^+} \frac{4}{3}(1 - t^{3/4}) = \frac{4}{3}(1-0) = \frac{4}{3}$$

converges

D. $\int_0^1 \ln t\,dt = \lim_{a\to0^+}\int_a^1 \ln t\,dt = \lim_{a\to0^+}\left(\left[t\ln t\right]_a^1 - \int_a^1 \frac{1}{t}t\,dt\right)$

$$\left(\begin{matrix} u=\ln t & v=t \\ du=\frac{1}{t} & dv=dt\end{matrix}\right)$$

$$= \lim_{a\to0^+}\left(1\cdot0 - a\cdot\ln a - \int_a^1 dt\right) = \lim_{a\to0^+} -a\ln a - (1-a)$$

$$= 0 - 1 + 0 = -1$$

how to do this? L'Hopital's rule!

diverges

E. $\displaystyle\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \Big[-\ln|1-x|\Big]_t^2$

$$= \lim_{t\to 1^+} -\ln 1 + \ln|1-t| = \lim_{t\to 1^+} \ln(t-1) = -\infty$$

diverges

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converges if $s>1$ (otherwise diverges)

F. $\displaystyle\int_0^\infty e^x e^{-sx}\,dx = \lim_{t\to\infty} \int_0^t e^{(1-s)x}\,dx = \lim_{t\to\infty} \left[\frac{e^{(1-s)x}}{1-s}\right]_0^t$

$s>1$

$$= \frac{1}{1-s} \lim_{t\to\infty} \left(e^{(1-s)t} - e^0\right) = \frac{1}{1-s}(0-1) = \frac{1}{s-1}$$

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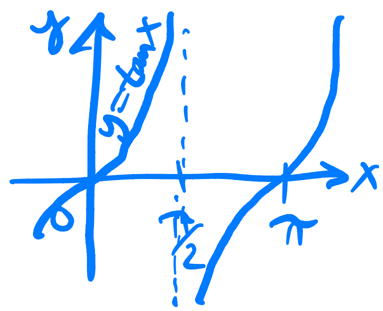
diverges

G. $\displaystyle\int_0^\pi \tan x\,dx = \int_0^{\pi/2} \tan x\,dx + \int_{\pi/2}^{\pi} \tan x\,dx$



$$\int_0^{\pi/2} \tan x\,dx = \lim_{t\to \frac{\pi}{2}^-} \int_0^t \frac{\sin x}{\cos x}\,dx$$

$$= \lim_{t\to \frac{\pi}{2}^-} \Big[-\ln|\cos x|\Big]_0^t = \lim_{t\to \frac{\pi}{2}^-} \Big(-\ln|\cos t| + \underbrace{\ln 1}_{0}\Big)$$

$$= +\infty$$

diverges

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converges

H. $\displaystyle\int_2^\infty \frac{dx}{x\ln^3 x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^3}$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3} = \lim_{t\to\infty} \left[-\frac{u^{-2}}{2}\right]_{\ln 2}^{\ln t}$$

$(u = \ln x)$

$$= \lim_{t\to\infty} \left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln t)^2}\right) = \frac{1}{2(\ln 2)^2} - 0 = \frac{1}{2(\ln 2)^2}$$