SOLUTIONS

Worksheet: Calculating Taylor series

The Taylor series *of* $f(x)$ *at basepoint* a *is*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots.$$

(*When* $a = 0$ *one calls it a* Maclaurin series, *but who cares really?*) *The* nth Taylor polynomial *is the partial sum of the series:*

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A. Compute the Taylor series of $f(x) = e^{3x}$ at $a = 0$. What is the interval of convergence?

$f(x) = e^{3x}$ $\quad f(0) = 1$

$f'(x) = 3e^{3x}$ $\quad f'(0) = 3$

$f''(x) = 3^2 e^{3x}$ $\quad \vdots$

$\vdots$

$f^{(n)}(x) = 3^n e^{3x}$ $\quad f^{(n)}(0) = 3^n$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$$

interval: $\rho = \lim_{n\to\infty} \dfrac{3^{n+1}|x|^{n+1}/(n+1)!}{3^n|x|^n/n!} = \lim_{n\to\infty} \dfrac{3^{n+1}|x|^{n+1}\, n!}{(n+1)\, n!\, 3^n |x|^n} = \lim_{n\to\infty} \dfrac{3|x|}{n+1} = 0 < 1$

$\Downarrow$

$$I = (-\infty, \infty)$$

B. Find $p_2(x)$ for $f(x) = \arctan(x)$ at $a = 0$.

$f(x) = \arctan x$ $\quad f(0) = 0$

$f'(x) = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -(1+x^2)^{-2}(2x)$ $\quad f''(0) = 0$

$= \dfrac{-2x}{(1+x^2)^2}$

$p_2(x) = 0 + 1\cdot x + \dfrac{0}{2}\cdot x^2$

$= x$

C. Compute the Taylor series of $f(x) = \sin x$ at $a = \pi$.

$f(x) = \sin x$ $\quad f(\pi) = 0$

$f'(x) = \cos x$ $\quad f'(\pi) = -1$

$f''(x) = -\sin x$ $\quad f''(\pi) = 0$

$f'''(x) = -\cos x$ $\quad f'''(\pi) = +1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(\pi) = 0$

$\vdots \qquad \vdots$

$$\sin x = 0 + (-1)(x-\pi) + \frac{0}{2}(x-\pi)^2 + \frac{(+1)}{3!}(x-\pi)^3 + \frac{0}{4!}(x-\pi)^4 + \frac{(-1)}{5!}(x-\pi)^5 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

D. Compute the Taylor series of $f(x) = \dfrac{1}{1+x}$ at $a = 0$. What is the interval of convergence? Confirm using your knowledge of geometric series.

$f(x) = (1+x)^{-1}$ $\quad f(0) = 1$

$f'(x) = -(1+x)^{-2}$ $\quad f'(0) = -1$

$f''(x) = +2(1+x)^{-3}$ $\quad f''(0) = +2$

$f'''(x) = -3!(1+x)^{-4}$ $\quad f'''(0) = -3!$

$f^{(4)}(x) = +4!(1+x)^{-5}$ $\quad f^{(4)}(0) = +4!$

$\vdots \qquad \vdots$

$f^{(n)}(0) = (-1)^n n!$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

check using geometric series:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \checkmark$$

interval: from geometric series, $|x| < 1$ so $I = (-1, 1)$