Math 252: Quiz 6 **3 March, 2022**

Name: SOLUTIONS

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30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [9 points] Compute and simplify the improper integrals, or show the integral diverges. Use correct limit notation.

a.

$$\int_4^\infty \frac{dx}{\sqrt[3]{x-2}} = \lim_{t\to\infty} \int_4^t (x-2)^{-1/3}\,dx \overset{(u=x-2)}{=} \lim_{t\to\infty} \int_2^{t-2} u^{-1/3}\,du$$

$$= \lim_{t\to\infty} \left[\frac{3}{2}u^{2/3}\right]_2^{t-2} = \lim_{t\to\infty} \frac{3}{2}\left((t-2)^{2/3} - 2^{2/3}\right)$$

$$= +\infty$$ diverges

b.

$$\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+} \int_t^1 x^{-1/3}\,dx = \lim_{t\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_t^1$$

$$= \frac{3}{2}\left(1 - \lim_{t\to 0^+} t^{2/3}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$$

c.

$$\int_{-\infty}^\infty \cos(x)\,dx = \int_{-\infty}^0 \cos(x)\,dx + \int_0^\infty \cos(x)\,dx$$

$$\int_0^\infty \cos x\,dx = \lim_{t\to\infty} \int_0^t \cos(x)\,dx$$

$$= \lim_{t\to\infty} \left[\sin(x)\right]_0^t = \lim_{t\to\infty} \sin t \quad \text{d.n.e.}$$

$\therefore$ integral diverges

2. [3 points] Find the general solution to the differential equation $y' = x^3$.

$$y(x) = \frac{1}{4}x^4 + C$$

3. [5 points] Verify that $y = \frac{2}{\sqrt{1-8x}}$ solves the differential equation $y' = y^3$.

$$y = 2(1-8x)^{-1/2}$$

$$y' = 2\left(-\frac{1}{2}\right)(1-8x)^{-3/2}(-8) = \frac{8}{(1-8x)^{3/2}}$$

$$y^3 = \left(\frac{2}{(1-8x)^{1/2}}\right)^3 = \frac{8}{(1-8x)^{3/2}}$$

$\checkmark$

4. [3 points] Find the particular solution to the differential equation $y' = 3x^2y$ that passes through $(0, 12)$, given that $y = Ce^{(x^3)}$ is the general solution.

$$12 = Ce^{0^3} = C$$

so $$y = 12e^{(x^3)}$$

5. [5 points] Suppose the region underneath $y = \sqrt{x}\,e^{-x/2}$, on the interval $[0, \infty)$, is rotated around the x-axis. Find the volume of the enclosed solid. (*Hint. Use correct and appropriate limit notation on the improper integral. Discs!*)

$$V = \int_0^\infty \pi\left(\sqrt{x}\,e^{-x/2}\right)^2 dx = \pi \int_0^\infty x e^{-x}\,dx$$

$$= \pi \lim_{t\to\infty}\left(\left[-xe^{-x}\right]_0^t - \int_0^t -e^{-x}\,dx\right)$$

$$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x}dx \end{pmatrix}$$

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} \overset{LH}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$$

$$= \pi \lim_{t\to\infty}\left(-te^{-t} + 0 + \int_0^t e^{-x}\,dx\right)$$

$$= \pi\left(0 + \lim_{t\to\infty}\left[-e^{-x}\right]_0^t\right)$$

$$= \pi \lim_{t\to\infty}\left(1 - e^{-t}\right) = \pi(1-0) = \pi$$

Extra Credit. [2 points] Consider any smooth curve $y = f(x)$ on the interval $[0,\infty)$. An improper integral computes the total length L of this curve. By comparing this integral to another improper integral, *one which you show is divergent*, explain why $L = +\infty$.

$$L = \int_0^\infty \sqrt{1 + f'(x)^2}\, dx$$

$$\geq \int_0^\infty \sqrt{1+0}\, dx = \lim_{t\to\infty} \int_0^t 1\, dx$$

compare!

$$= \lim_{t\to\infty} t = +\infty$$

So length integral diverges

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