Math 252: Quiz 5 SOLUTIONS **10 February, 2022**

Name: ______________________ ______ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [4 points] Compute and simplify the definite integral:

$$\int_{-1}^{0} x e^x \, dx = xe^x \Big]_{-1}^{0} - \int_{-1}^{0} e^x \, dx$$

$$\left(\begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x dx \end{array} \right)$$

$$= 0 - (-1)e^{-1} - \left[e^x\right]_{-1}^{0} = +e^{-1} - e^0 + e^{-1}$$

$$= \frac{2}{e} - 1$$

2. [4 points] Compute and simplify the indefinite integral:

$$\int \cos^3\theta \sin^3\theta \, d\theta = \int \cos^3\theta \underbrace{\sin^2\theta}_{=1-\cos^2\theta} \sin\theta \, d\theta$$

$$= \int \cos^3\theta (1 - \cos^2\theta) \sin\theta \, d\theta$$

$$\left(\begin{array}{l} u = \cos\theta \\ du = -\sin\theta \, d\theta \end{array} \right)$$

$$= \int u^3 (1 - u^2)(-du) = \int u^5 - u^3 \, du$$

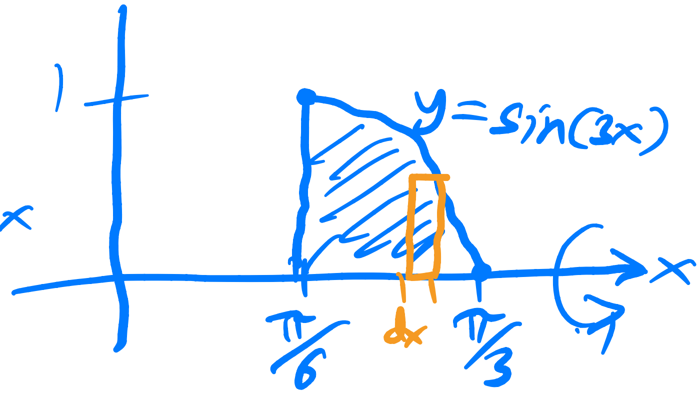
$$= \frac{1}{6}\cos^6\theta - \frac{1}{4}\cos^4\theta + C$$

See alternate (also correct) solution at end.

3. [5 points] Sketch the region between $y = \sin(3x)$ and the x-axis on $\pi/6 \le x \le \pi/3$. Then compute (and simplify) the volume of the solid formed by rotating this region around the x-axis.

discs:

$$V = \int_{\pi/6}^{\pi/3} \pi \sin^2(3x)\, dx$$



$$= \frac{\pi}{2} \int_{\pi/6}^{\pi/3} 1 - \cos(6x)\, dx$$

(use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$)

$$= \frac{\pi}{2} \left[x - \frac{\sin(6x)}{6} \right]_{\pi/6}^{\pi/3}$$

$$= \frac{\pi}{2} \left[\left(\frac{\pi}{3} - 0\right) - \left(\frac{\pi}{6} - 0\right) \right] = \frac{\pi^2}{12}$$

4. [4 points] Compute and simplify the indefinite integral:

$$\int x^2 \ln x\, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x}\, dx$$

$$\begin{pmatrix} u = \ln x & v = x^3/3 \\ du = \frac{1}{x} dx & dv = x^2 dx \end{pmatrix}$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + C$$

$$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C$$

5. [4 points] Compute and simplify the indefinite integral:

$$\int \sec t\,dt = \int \sec t \,\frac{\sec t + \tan t}{\sec t + \tan t}\,dt$$

$$= \int \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t}\,dt \qquad \begin{cases} u = \sec t + \tan t \\ du = \sec t \tan t + \sec^2 t\,dt \end{cases}$$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec t + \tan t| + C$$

6. [4 points] Compute and simplify the indefinite integral:

$$\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx = \int \tan x(\sec^2 x - 1)\,dx$$

$$(\tan^2 x = \sec^2 x - 1)$$

$$= \int \tan x \sec^2 x\,dx - \int \frac{\sin x}{\cos x}\,dx$$

$$= \int u\,du - \int \frac{-dw}{w} = \frac{1}{2}u^2 + \ln|w| + c$$

$$\begin{pmatrix} u = \tan x & & w = \cos x \\ du = \sec^2 x\,dx & \text{and} & -dw = \sin x\,dx \end{pmatrix}$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$

See alternate (also correct) solution at end.

see text in section 3.2

Extra Credit. [2 points] Compute and simplify the indefinite integral:

$$\int \sec^3 x\,dx = \tan x \sec x - \int \tan x \sec x \tan x\,dx$$

$u = \sec x \quad v = \tan x$

$\left(du = \sec x \tan x\,dx \quad dv = \sec^2 x\,dx\right)$

$$= \tan x \sec x - \int (\sec^2 x - 1)\sec x\,dx$$

$(\tan^2 x = \sec^2 x - 1)$

$$= \tan x \sec x + \int \sec x\,dx - \int \sec^3 x\,dx$$

$$= \tan x \sec x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx$$

so

$$A = \tan x \sec x + \ln|\sec x + \tan x| - A$$

so $\quad 2A = \ldots$

so: $\displaystyle\int \sec^3 x\,dx = \frac{1}{2}\left(\tan x \sec x + \ln|\sec x + \tan x|\right) + C$

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2. alternate solution

$$\int \cos^3\theta \sin^3\theta \, d\theta$$

$$= \int \underbrace{\cos^2\theta}_{=1-\sin^2\theta} \sin^3\theta \cos\theta \, d\theta$$

$$= \int (1-\sin^2\theta)\sin^3\theta \cos\theta \, d\theta$$

$(u = \sin\theta, \; du = \cos\theta \, d\theta)$

$$= \int (1-u^2)u^3 \, du$$

$$= \int u^3 - u^5 \, du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4\theta - \frac{1}{6}\sin^6\theta + C$$

this is the same function up to an additive constant (i.e. different "$+C$")

because $\sin^2\theta = 1-\cos^2\theta$

6. alternate solution

$$\int \tan^3 x\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,dx$$

$$= \int \frac{\sin^2 x}{\cos^3 x}\sin x\,dx$$

$$= \int \frac{1-\cos^2 x}{\cos^3 x}\sin x\,dx$$

$\leftarrow \begin{cases} u = \cos x \\ du = -\sin x\,dx \end{cases}$

$$= \int \frac{1-u^2}{u^3}(-du) = \int \frac{1}{u} - u^{-3}\,du$$

$$= \ln|u| + \frac{1}{2}u^{-2} + C$$

$$= \ln|\cos x| + \frac{1}{2}\frac{1}{\cos^2 x} + C$$

same up to an additive constant because

$$\tan^2 x = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$$