Math 252: Quiz 10 SOLUTIONS **14 April, 2022**

Name: ______________________ | ________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [8 points] Using any convenient method, write the Maclaurin series of the given function. Use sigma notation for your answer.

a. $g(x) = xe^{-2x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ ← familiar!

$$xe^{-2x} = \sum_{n=0}^{\infty} x\frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{n+1}}{n!}$$

b. $f(x) = \dfrac{\sin x}{x}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ ← familiar!

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

2. [4 points] Use the answer from problem **1 b** on the previous page to compute the integral:

$$\int_0^x \frac{\sin t}{t}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}\,dt$$

now integrate term-by-term

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{t^{2n+1}}{2n+1} \right]_0^x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)}}$$

3. [4 points] Use the ratio (or root) test, plus a check on series convergence at the endpoints, to show that the interval of convergence of the following familiar Maclaurin series is $[-1,1]$.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\rho = \lim_{n\to\infty} \frac{\frac{|x|^{2n+3}}{2n+3}}{\frac{|x|^{2n+1}}{2n+1}} = \lim_{n\to\infty} |x|^2 \frac{2n+1}{2n+3}$$

$$= |x|^2 \lim_{n\to\infty} \frac{2n+1}{2n+3} = |x|^2 \cdot 1 = |x|^2 < 1$$

L'Hopital's or other method...

$\underline{x=-1:}$ $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(-1)}{2n+1}$ $\leftarrow$ AST. shows converge

$\underline{x=1:}$ $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ $\leftarrow$ AST shows converge

interval $[-1,1]$

4. [5 points] Use the binomial series for $(1+x)^r$ to write out the first four nonzero terms of

$$f(x) = (1+x^2)^{1/3}$$

(**Hint.** This means you will write the Taylor polynomial of degree 6 for $f(x)$.)

$$(1+x^2)^{1/3} = \underbrace{1 + \overset{\binom{1/3}{1}}{\frac{1}{3}}x^2 + \overset{\binom{1/3}{2}}{\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}}x^4 + \overset{\binom{1/3}{3}}{\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}}x^6}_{\text{first four nonzero terms}} + \dots$$

simplified form: $\Leftarrow$ not required

$$P_6(x) = 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6$$

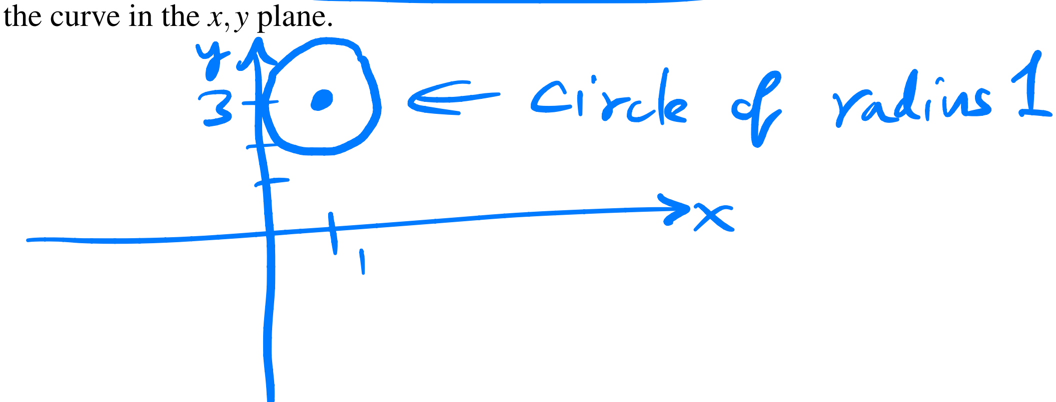
5. [4 points] Consider the parametric curve $x = 1+\cos t$, $y = 3-\sin t$.

a. Eliminate the parameter to convert into rectangular form.

$x-1 = \cos t$, $y-3 = -\sin t$ $\therefore$

$$(x-1)^2 + (y-3)^2 = 1$$

b. Sketch the curve in the x, y plane.



Extra Credit. [2 points] Finding the antiderivative

$$\int \sqrt{x}\,e^x\,dx$$

is traditionally regarded as impossible. Indeed it is impossible if you want a finite expression in terms of familiar functions, but fairly easy if you accept a series, one which is not quite a standard power series, for the answer. Starting from the familiar Maclaurin series for e^x, find a series form of this antiderivative.

$$\sqrt{x}\,e^x = x^{1/2}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1/2}}{n!}$$

not standard power series because not whole numbers in powers

$$\int \sqrt{x}\,e^x\,dx = C + \sum_{n=0}^{\infty}\frac{x^{n+3/2}}{(n+\frac{3}{2})\,n!}$$

remember this

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