Math 252: Quiz 8 SOLUTIONS **3 November, 2022**

Name: ______________________ **/ 25**

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [9 points] Use either the **comparison test** or the **limit comparison test** to determine whether the series converge or diverge. Please show enough work to demonstrate how the test is applied.

a. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ (note $\frac{1}{2n+1} \leq \frac{1}{2n}$ ... wrong side for comparison)

limit compare to $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic)

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{2n}{2n+1} \overset{LH}{=} \lim_{n\to\infty} \frac{2}{2} = 1 \neq 0, \infty$$

so diverges

b. $\sum_{n=2}^{\infty} \frac{1}{n^2+\ln n}$ $\ln n \geq 0$ so $n^2 + \ln n \geq n^2$

so $\frac{1}{n^2+\ln n} \leq \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p=2)$

so converges by comparison test

[limit comparison is possible, but hard to use]

c. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ $\sin^2 n \leq 1$ so $\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is geometric, $r = \frac{1}{2} < 1$, converges.

so converges by comparison test

2. [4 points] Apply the **integral test** to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

$p \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_1^t$$

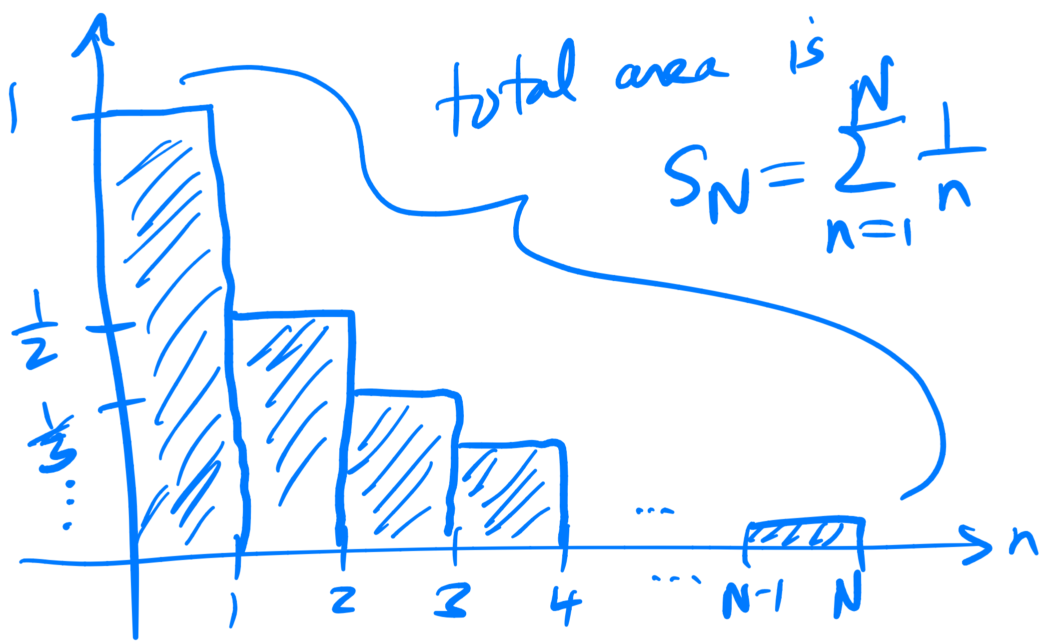
$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$$

$p > 1$

which is finite (improper integral converges)

so series converges

3. [3 points] Sketch a partial sum S_N of the harmonic series, as the total area of rectangles of width one. (**Please label axes appropriately.**)



4. [9 points] Use an appropriate test to determine whether the series converge or diverge. Please state what test you are using, and show enough work to demonstrate how the test is applied.

a. $\sum_{n=2}^{\infty} \frac{(n-2)!}{n!}$

$$\frac{(n-2)!}{n!} = \frac{\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{2}\cdot\cancel{1}}{n(n-1)\cancel{(n-2)}\cdots\cancel{2}\cancel{1}} = \frac{1}{n(n-1)}$$

limit compare to $\sum_{n=2}^{\infty} \frac{1}{n^2}$: $\lim_{n\to\infty} \frac{\frac{1}{n(n-1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} \overset{LH}{=} \frac{2}{2} = 1 \neq 0, \infty$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2$), so converges

[also can use telescoping or integral test on $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$]

b. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ integral test:

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^2} \quad \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right]$$

$$= \lim_{t\to\infty} \left[-u^{-1}\right]_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln t} = \frac{1}{\ln 2} - 0$$

(finite)

so series converges

c. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^3+1}}$

divergence test: $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n} + \frac{1}{n^4}}} \overset{\text{“}\frac{1}{0}\text{”}}{=} +\infty \neq 0$$

so diverges

[also: integral test, limit compare to $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^3}} = \sum_{n=1}^{\infty} n^{1/2}$]

EC. [1 points] (**Extra Credit**) My computer says that the 100th partial sum of the series $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is $S_{100} = 1.2020074$. How accurate is this? Use an integral to estimate the size of the remainder $R_{100} = S - S_{100}$.

$$R_{100} \leq \int_{100}^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \left[\frac{-x^{-2}}{2} \right]_{100}^{t}$$

$$= \lim_{t\to\infty} \frac{(100)^{-2}}{2} - \frac{t^{-2}}{2} = \frac{1}{2\cdot(100)^2} = 5\times 10^{-5}$$

so $S_{100} = \sum_{n=1}^{100} \frac{1}{n^3}$ is within 5×10^{-5} of $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$

so $S = \underbrace{1.2020}_{\text{these are correct}}\underbrace{\ldots}_{}$ ← unknown digits

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