Math 252: Quiz 4 **22 September, 2022**

Name: SOLUTIONS

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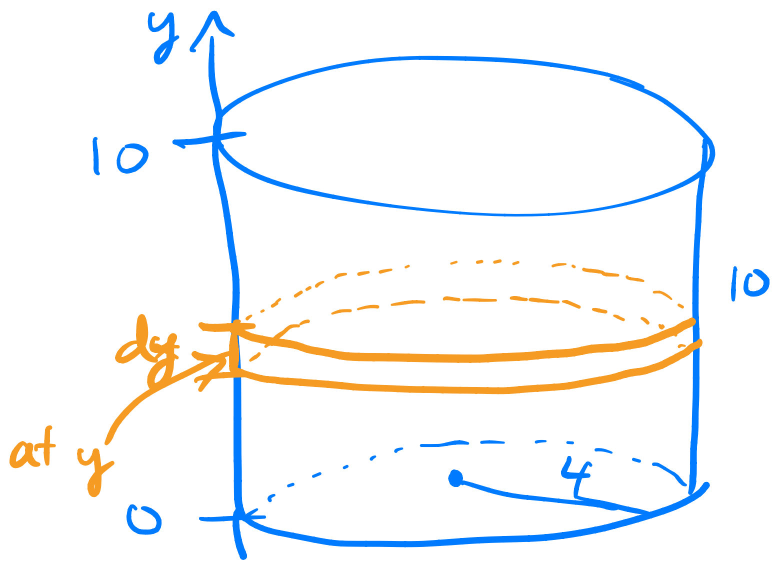
30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [10 points] Find the work required to pump all the water out of the top of a cylinder which has a circular base of radius 4 meters and height 10 meters. Use the fact that water has a mass density of 1000 $\mathrm{kg/m^3}$, and use $g = 10\ \mathrm{m/s^2}$ as an approximation of the acceleration of gravity. (*Hint: Start by drawing a decent sketch and considering a slice of water; a good sketch is worth 2 points. Simplify your answer and give units.*)



for a slice:

$$dV = \pi \cdot 4^2 \cdot dy$$
$$= 16\pi\, dy \quad [m^3]$$

$$dm = 16\pi\rho\, dy \quad [kg]$$

$$dW = (\text{force})\cdot(\text{distance})$$
$$= \underbrace{16\pi\rho\, dy\, g}_{F=mg} \cdot (10-y)$$

so

$$W = \int_0^{10} 16\pi\rho g\, dy\, (10-y)$$
$$= 16\pi \cdot 1000 \cdot 10 \int_0^{10} 10-y\, dy$$
$$= 16\pi\, 10^4 \left[10y - \frac{y^2}{2}\right]_0^{10} \qquad [J = Nm]$$
$$= 16\pi\, 10^4 \left(100 - \frac{100}{2}\right) = 16 \cdot 50 \cdot \pi \cdot 10^4 \quad \checkmark$$
$$= 8\pi \times 10^6\ J$$

2. [8 points] Find the derivative $\frac{dy}{dx}$ or the indefinite integral. (*Hint: Use "+C" where needed.*)

a. $y = \log_{10} x$ $= \dfrac{\ln x}{\ln 10}$

$$\frac{dy}{dx} = \frac{1}{x \ln 10}$$

b. $\displaystyle\int \frac{(\ln x)^2\,dx}{x} =$ $\displaystyle\int u^2\,du = \frac{u^3}{3} + C$

$\left[\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array}\right]$

$$= \frac{(\ln x)^3}{3} + C$$

c. $y = x^{(ex)}$

$$\ln y = ex \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = e\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$$

$$= e(\ln x + 1)$$

$$\frac{dy}{dx} = x^{(ex)} e(\ln x + 1)$$

d. $y = \ln\left(\dfrac{x+a}{x-a}\right)$ $= \ln(x+a) - \ln(x-a)$

$$\frac{dy}{dx} = \frac{1}{x+a} - \frac{1}{x-a} = \frac{-2a}{x^2 - a^2}$$

just fine to rewrite!

3. [7 points] A 1 meter car antenna has linear mass density, starting from the base at $x = 0$, of $\rho(x) = 2 + \dfrac{x}{100}$ grams per centimeter. What is its mass? Simplify your answer and give units.

$$m = \int_0^{100} \underbrace{\left(2 + \frac{x}{100}\right)}_{\left[\frac{g}{cm}\right]} \underbrace{dx}_{[cm]}$$

$$= 2x + \frac{x^2}{200} \Big]_0^{100}$$

$$= 200 + \frac{100^2}{200} = 200 + \frac{100}{2} = 250\,g$$

EC. [1 points] (Extra Credit) Assuming $x > 0$, fully simplify:

$$\frac{d}{dx}\left(\int_x^{x^2} \frac{dt}{t}\right) \overset{①}{=} \frac{d}{dx}\left(\ln|t|\Big]_x^{x^2}\right) = \frac{d}{dx}\left(\ln(x^2) - \ln x\right)$$

$$= \frac{1}{x^2}\cdot 2x - \frac{1}{x} = \frac{2}{x} - \frac{1}{x}$$

$$= \frac{1}{x}$$

$$\boxed{\begin{array}{l} ① = \text{FTC II} \\ ② = \text{FTC I} \end{array}}$$

$$\frac{d}{dx}\left(\int_x^{x^2}\frac{dt}{t}\right) \overset{②}{=} \frac{d}{dx}\left(\int_x^a \frac{dt}{t} + \int_a^{x^2}\frac{dt}{t}\right) = \frac{d}{dx}\left(-\int_a^x \frac{dt}{t} + \int_a^{x^2}\frac{dt}{t}\right)$$

$$= -\frac{1}{x} + \frac{1}{x^2}\cdot 2x = \frac{1}{x}$$

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