Math 252: Quiz 10 SOLUTIONS **1 December, 2022**

Name: ______________________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [8 points] Using any convenient method, find the Maclaurin series of the given function.

a. $f(x) = 7^x$ $= \left(e^{\ln 7}\right)^x = e^{(\ln 7)x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{((\ln 7)x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 7)^n x^n}{n!}$$

b. $f(x) = \cos(\sqrt{x})$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$f(x) = \cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

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2. [4 points] Using the answer from **1 b**, express the integral as an infinite series.

$\int \cos(\sqrt{x})\,dx =$ $\int \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{n+1}}{n+1} + C$

$= C + x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 4!} - \frac{x^4}{4\cdot 6!} + \dots$

either is fine

3. [6 points] Let $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials, of degrees 1 and 2, of $f(x)$ at $x = 2$.

$a=2$

$f(x) = x^{1/3}$ $\qquad f(2) = 2^{1/3}$ $\qquad \therefore$ $p_1(x) = 2^{1/3} + \frac{1}{3\cdot 2^{2/3}}(x-2)$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(2) = \frac{1}{3\cdot 2^{2/3}}$

$f''(x) = \frac{-2}{9}x^{-5/3}$ $\qquad f''(2) = \frac{-2}{9\cdot 2^{5/3}}$

$p_2(x) = p_1(x) - \frac{2}{9\cdot 2^{5/3}}\frac{1}{2}(x-2)^2$

b. Use the first Taylor polynomial to estimate $\sqrt[3]{3}$.

$\sqrt[3]{3} = f(3) \approx p_1(3) = 2^{1/3} + \frac{1}{3\cdot 2^{2/3}}(3-2)$

$= 2^{1/3} + \frac{1}{3\cdot 2^{2/3}}$

calculator:

$\sqrt[3]{3} = 1.4422$

$p_1(3) = 1.4699$ (also: $p_2(3) = 1.4349$)

4. [7 points] Use the ratio or root test, plus a check on series convergence at the endpoints, to find the interval of convergence of the Maclaurin series for $f(x) = \ln(1+x)$.

$$f(x) = \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

ratio test:

$$\rho = \lim_{n\to\infty} \frac{\left|(-1)^{n+1} \frac{x^{n+2}}{n+2}\right|}{\left|(-1)^n \frac{x^{n+1}}{n+1}\right|} = \lim_{n\to\infty} \frac{|x|^{n+2}\,(n+1)}{(n+2)\,|x|^{n+1}}$$

$$= |x| \lim_{n\to\infty} \frac{n+1}{n+2} \overset{L'H}{=} |x| \cdot 1 < 1$$

$$-1 < x < 1$$

$x = -1$: $$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$$

$$= -\sum_{m=1}^{\infty} \frac{1}{m}$$

diverges: harmonic series

$x = 1$: $$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$ alternating

$b_n = \frac{1}{n+1} \to 0$, decrease

$\therefore$ converges by AST

interval of convergence: $I = (-1, 1]$

EC. [1 points] (**Extra Credit**) Find the value of the 24th derivative of $f(x) = e^{x^2}$ at $x = 0$. (*Hint. Taylor series? It relates to derivatives at $x = 0$.*)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \therefore f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

but: $c_n = \dfrac{f^{(n)}(0)}{n!}$ is the coefficient of x^n

so: $\dfrac{1}{12!} = \dfrac{f^{(24)}(0)}{24!}$ so: $f^{(24)}(0) = \dfrac{24!}{12!}$

$= 24 \cdot 23 \cdot \cdots 14 \cdot 13$

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