Your Name

SOLUTIONS

Signature (you agree to complete honestly)

Student ID #

Start Time

End Time

Page	Total Points	Score
2	10	
3	10	
4	11	
5	13	
6	10	
7	8	
8	7	
9	8	
10	10	
11	10	
12	3	
Total	100	

- You will have 2.5 hours to complete the exam.
- This test is closed book and you may not use a calculator.
- You may use one side of a single piece of paper (8 1/2 in. x 11 in.) of handwritten notes.
- In order to receive full credit (or partial credit in the case of incorrect solutions), you must **show your work.** Please write out your computations on the exam paper.
- Simplify all obvious expressions.
- **PLACE A BOX AROUND** **YOUR FINAL ANSWER** **to each question** where appropriate.

1. (20 points) Evaluate the following integrals.

(a) $\int_0^{\pi/4} \sec^4 x \tan^2 x \, dx$ $= \int_0^{\pi/4} \underbrace{\sec^2 x}_{= 1+\tan^2 x} \tan^2 x \sec^2 x \, dx$

$= \int_0^{\pi/4} (1+\tan^2 x)\tan^2 x \sec^2 x \, dx$

$\leftarrow \begin{cases} u = \tan x \\ du = \sec^2 x \, dx \end{cases}$

$= \int_0^1 (1+u^2)u^2 \, du$

$= \left[\frac{u^3}{3} + \frac{u^5}{5}\right]_0^1 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

(b) $\int \arctan(2x) \, dx$

$\leftarrow \begin{cases} u = \arctan(2x) & v = x \\ du = \frac{1}{1+(2x)^2} \cdot 2dx = \frac{2}{1+4x^2} dx & dv = dx \end{cases}$

$= x\arctan(2x) - \int \frac{2x}{1+4x^2} dx$

$\leftarrow \begin{cases} w = 1+4x^2 \\ dw = 8x \, dx \\ \frac{1}{4}dw = 2x \, dx \end{cases}$

$= x\arctan(2x) - \int \frac{\frac{1}{4}dw}{w}$

$= x\arctan(2x) - \frac{1}{4}\ln|w| + C$

$= x\arctan(2x) - \frac{1}{4}\ln(1+4x^2) + C$

(c) $\int \frac{4x}{(x^2+4)(x-2)}\,dx$

$$\frac{4x}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$$

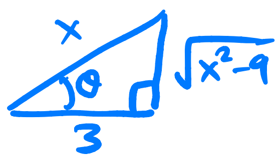
$$0x^2+4x+0 = (Ax+B)(x-2) + C(x^2+4)$$

$$= (A+C)x^2 + (-2A+B)x + (-2B+4C)$$

$$\left.\begin{array}{l} \left.\begin{array}{l} A+C=0 \\ -2A+B=4 \end{array}\right\} 2C+B=4 \\ -2B+4C=0 \ \} \ 2C-B=0 \end{array}\right\} 4C=4 \Rightarrow C=1,\ A=-1,\ B=2$$

$$\int \cdots dx = \int \frac{-x}{x^2+4} + \frac{2}{x^2+4} + \frac{1}{x-2}\,dx = -\frac{1}{2}\ln(x^2+4) + \arctan(x/4) + \ln|x-2| + C$$

(d) $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$



$$\begin{cases} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\,d\theta \end{cases}$$

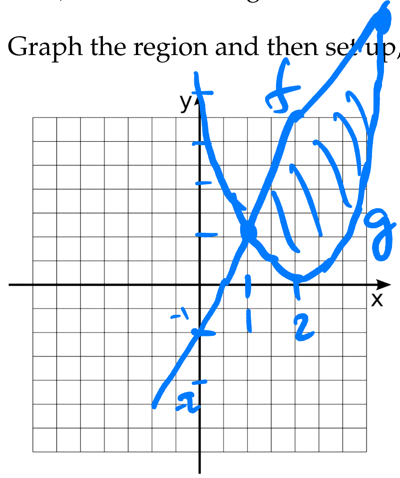
$\sec^2\theta - 1 = \tan^2\theta$

$$= \int \frac{3\sec\theta\tan\theta\,d\theta}{3^2\sec^2\theta\sqrt{9\sec^2\theta-9}} = \int \frac{\cancel{3}\,\cancel{\sec\theta}\,\cancel{\tan\theta}}{3^{\cancel{2}}\sec^{\cancel{2}}\theta\,3\,\cancel{\tan\theta}}\,d\theta$$

$$= \frac{1}{3^2}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

2. (11 points) Let R be the region bounded by the graphs of $f(x) = 2x - 1$, $g(x) = (x-2)^2$.

(a) Graph the region and then set up, but do not solve, an integral that gives the **area** of R.



$2x-1=(x-2)^2$

$2x-1=x^2-4x+4$

$x^2-6x+5=0$

$(x-1)(x-5)=0$

$x=1,5$

$A=\int_1^5 (2x-1)-(x-2)^2\,dx$

(b) Set up, but do not solve, an integral that finds the **volume** of the solid when R is rotated about the y-axis.

Shells:

$V=\int_1^5 2\pi x\left(2x-1-(x-2)^2\right)dx$

(c) Set up, but do not solve, an integral that finds the volume of the solid when R is rotated about the line $y=-1$.

Washers:

$$V=\int_1^5 \pi\left((2x-1+1)^2-\left((x-2)^2+1\right)^2\right)dx$$

in fact I won't ask this

(d) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square whose sides are the length of the base in the region R. Set up, but do not solve, an integral that gives the volume of this solid.

$$V=\int_1^5 \left((2x-1)-(x-2)^2\right)^2 dx$$

3. (4 points) Let $a_n = \sin\left(\dfrac{n - \pi n^2}{2n^2+3}\right)$.

(a) Determine whether the sequence a_n converges. If it is convergent determine what it converges to.

$$\lim_{n\to\infty} \sin\left(\frac{n-\pi n^2}{2n^2+3}\right) = \sin\left(\lim_{n\to\infty}\frac{-\pi n^2+n}{2n^2+3}\right) = \sin\left(-\frac{\pi}{2}\right)$$

(Converges) $= -1$

(b) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. Justify your answer.

by divergence test, this series diverges

4. (5 points) Find the sum of the following series exactly.

a) $\sum_{n=1}^{\infty} (-2)^n 2^{-2n+1}$ geometric

$$= (-2)2^{-1} + (-2)^2 2^{-3} + (-2)^3 2^{-5} + \dots$$

$$= -1 + \frac{1}{2} - \frac{1}{2^2} + \dots$$

$a = -1$, $r = -1/2$ $\therefore$

$$\sum \dots = \frac{-1}{1+\frac{1}{2}} = -\frac{2}{3}$$

b) $\sum_{n=0}^{\infty} \dfrac{3(-1)^n 2^n}{n!}$ $\leftarrow 3e^x$ for some x

$$= 3\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$$

$$= 3e^{-2} = \frac{3}{e^2}$$

5. (4 points) Find the Taylor series for the function $f(x) = e^{3x}$ centered at the point $a = -1$. Give your answer in summation notation.

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$f''(x) = 3^2 e^{3x}$

$\vdots$

$f^{(n)}(x) = 3^n e^{3x}$

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{3^n e^{-3}}{n!}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{3^n e^{-3}}{n!}(x+1)^n}$$

$$e^{3x} = e^{3(x+1)} e^{-3}$$

$$= e^{-3}\sum_{n=0}^{\infty}\frac{(3(x+1))^n}{n!}$$

$= [\text{same}]$

alt. method

6. (7 points)

(a) Determine whether the improper integral $\int_1^\infty xe^{-x^2}\,dx$ converges or diverges. Evaluate it if it is convergent.

$$\int \cdots dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}dx \qquad \begin{cases} u = x^2 \\ \frac{du}{2} = x\,dx \end{cases}$$

$$= \lim_{t\to\infty} \int_1^{t^2} e^{-u}\frac{du}{2} = \frac{1}{2}\lim_{t\to\infty}\left[-e^{-u}\right]_1^{t^2}$$

$$= \frac{1}{2}\lim_{t\to\infty}\left(e^{-1} - e^{-t^2}\right) = \frac{1}{2}(e^{-1} - 0) = \frac{1}{2e}$$

(b) Use the integral test, and your answer from (a), determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. You must explicitly verify that the integral test applies to this series. No credit will be given if another test is used.

$f(x) = xe^{-x^2} \geq 0$

decreasing on $[1, \infty)$

$f(n) = a_n = ne^{-n^2}$

integral test shows $\sum_{n=1}^{\infty} ne^{-n^2}$ converges

7. (3 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. $= -\frac{1}{1} + \frac{1}{4} - \frac{1}{9} + \cdots$ $b_n = \frac{1}{n^2}$

(a) Find s_4. No need to simplify.

$$s_4 = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16}$$

(b) At most, how far is this from the actual sum? I.e., what is the |error|?

$$|\text{error}| = |R_4| \leq b_5 = \frac{1}{5^2} = \frac{1}{25}$$

$$R_4 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - S_4$$

8. (8 points) Determine whether the following series converge or diverge. You must clearly explain your reasoning.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{4+2n^3}$

compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $p=2$ $\therefore$ converge

limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+3}}{4+2n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2\sqrt{n^2+3}}{2n^3+4} = \lim_{n\to\infty} \frac{\sqrt{n^6+3n^4}}{2n^3+4}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+3/n^2}}{2+4/n^3} = \frac{1}{2} \neq 0, \infty$$

$\therefore$ both converge

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n+3}$

$$\left.\begin{array}{l} b_n = \frac{2}{n+3} \geq 0 \\ b_n \text{ decreases} \\ \lim_{n\to\infty} b_n = 0 \end{array}\right\}$$

AST shows converges

9. (7 points) Find the center, radius of convergence, and the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{(n+1)^n}$

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{|2x+3|^n}{(n+1)^n}} = \lim_{n\to\infty} \frac{|2x+3|}{n+1} = 0 < 1$$

so

$$I = (-\infty, \infty)$$

$$R = \infty$$

[center meaningless]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n \cdot n^2}$

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{|x-2|^n}{3^n n^2}} = \lim_{n\to\infty} \frac{|x-2|}{3(\sqrt[n]{n})^2}$$

$$= \frac{|x-2|}{3} < 1 \iff -3 < x-2 < 3 \iff -1 < x < 5$$

$x=-1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p=2$

$x=5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges AST

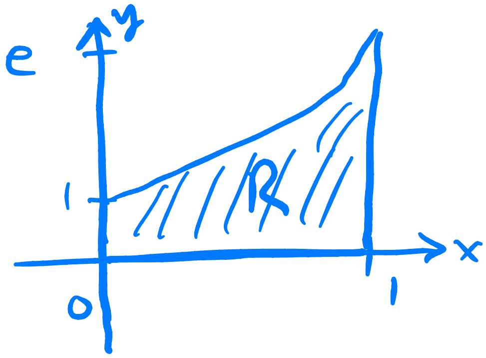
$R = 3$

$I = [-1, 5]$

center $= 2$

10. (8 points) Let $\mathcal{R}$ be the region bounded by $y = e^x$ and $y = 0$, $0 \le x \le 1$.

(a) Sketch the region and find the area of $\mathcal{R}$.



$$A = \int_0^1 e^x - 0\,dx = e^x\Big]_0^1$$

$$= e - 1$$

(b) Find the centroid of the the region $\mathcal{R}$.

$$m = \int_0^1 e^x - 0\,dx = e - 1$$

density $\rho = 1$

I won't ask this

$$M_y = \int_0^1 x e^x\,dx = x e^x\Big]_0^1 - \int_0^1 e^x\,dx$$

$$\begin{pmatrix} u = x & v = e^x \\ du = dx & dv = e^x dx \end{pmatrix}$$

$$= e - (e - 1) = 1$$

$$M_x = \int_0^1 \left(\tfrac{1}{2}e^x\right) e^x\,dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{2}\,\frac{e^{2x}}{2}\Big]_0^1$$

$$= \frac{1}{4}(e^2 - 1)$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{e-1}, \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{4}(e^2-1)}{e-1}$$

11. (10 points) Consider $x = t + 2\ln t$, $y = t - \ln t$. $\}$ only valid for $t > 0$

(a) Find and simplify $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1-\frac{1}{t}}{1+\frac{2}{t}} = \frac{t-1}{t+2}$$

(b) Determine the location of any horizontal tangents. If none exist, explain why.

$$m=0 \iff \frac{dy}{dt}=0 \iff 1-\frac{1}{t}=0 \iff t=1 \Rightarrow (x,y)=(1,1)$$

(c) Find and simplify $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\left(\frac{t-1}{t+2}\right)'}{1+\frac{2}{t}} = \frac{\frac{1(t+2)-(t-1)\cdot 1}{(t+2)^2}}{1+\frac{2}{t}}$$

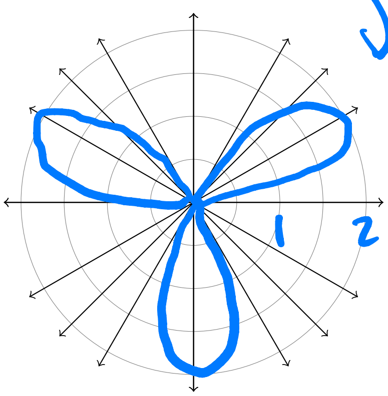
$$= \frac{t+2-t+1}{(t+2)^2}\,\frac{t}{t+2} = \frac{3t}{(t+2)^3}$$

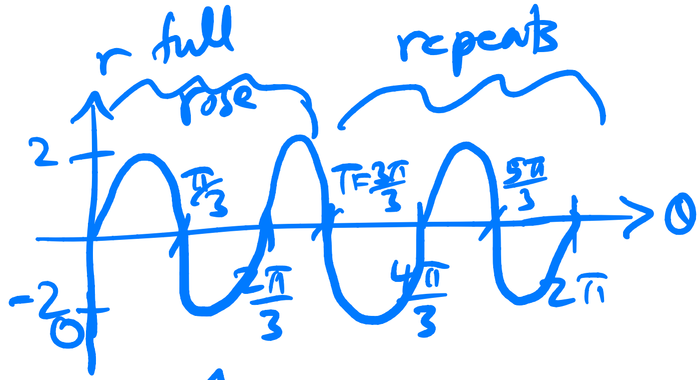
(d) Determine the values of t for which the curve is concave up.

$$\frac{d^2y}{dx^2} > 0 \quad \text{for all } t > 0$$

12. (10 points) Consider the curve $r = 2\sin(3\theta)$.

(a) Sketch the curve $r = 2\sin(3\theta)$.





notes to myself

(b) Find the area enclosed by one petal.

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin(3\theta))^2 \, d\theta$$

$$= 2\int_0^{\pi/3} \sin^2(3\theta)\, d\theta$$

$$= \int_0^{\pi/3} 1 - \cos(6\theta)\, d\theta = \left[\theta - \frac{\sin(6\theta)}{6}\right]_0^{\pi/3}$$

$$= \frac{\pi}{3} \approx 1$$

(c) Set up, but do not solve, an integral that gives the length of the polar curve traced out once.

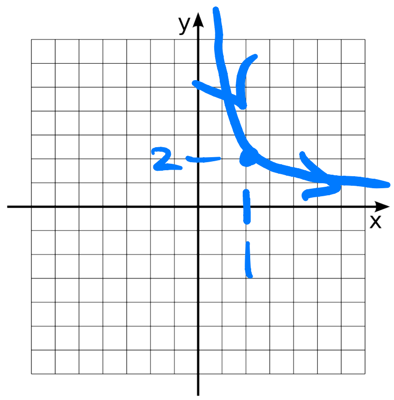
$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \frac{dr}{d\theta} = 6\cos(3\theta)$$

$$= \int_0^{\pi} \sqrt{4\sin^2(3\theta) + 36\cos^2(3\theta)}\, d\theta$$

13. (3 points) Consider the curve defined by the parametric equations $x = e^t$, $y = 2e^{-t}$.

(a) Graph the curve and indicate with an arrow the direction in which the curve is traced as t increases and (b) eliminate the parameter to find a Cartesian equation of the curve. [Make sure to specify any restriction on the variables.]

(a)



$x = e^t$

$y = \frac{2}{e^t} = \frac{2}{x}$

(b) $y = \frac{2}{x}$

Extra Credit (3 points) Find the length of the polar curve $r = e^{-\theta}$ for $\theta \geq 0$.

fun: plot on desmos and blow up near origin

$r = e^{-\theta}$

$\frac{dr}{d\theta} = -e^{-\theta}$

$$L = \int_0^\infty \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta$$

$$= \int_0^\infty \sqrt{2e^{-2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^\infty e^{-\theta}\, d\theta = \sqrt{2} \lim_{t\to\infty} \left[-e^{-\theta}\right]_0^t$$

$$= \sqrt{2} \lim_{t\to\infty} \left[1 - e^{-t}\right] = \sqrt{2} \cdot 1 = \sqrt{2}$$

my E.C. likely to be harder/trickier