Name: ____________________

Math 252 Calculus 2 (Bueler) Tuesday, 13 December 2022

Final Exam

No book, electronics, calculator, or internet access. 125 points possible. 125 minutes maximum. Allowed notes: 1/2 sheet of letter paper (i.e. half of 8.5×11 sheet), with anything written on both sides.

1. (a) (*5 pts*) Sketch the region bounded by $y = \dfrac{1}{x}$ and the x-axis, on the interval $1 \le x < +\infty$.

(b) (*8 pts*) Compute the volume of the solid of revolution found by rotating the region in **(a)** around the x-axis, or show that it is infinite.

2. (*7 pts*) Set up, but **do not evaluate**, an integral for the arclength of the parametric curve defined by $x(t) = e^t$, $y(t) = \ln t$ on the interval $1 \le t \le 10$.

3. (*8 pts*) Compute the indefinite integral:

$$\int x\, 5^{x^2}\, dx =$$

4. Evaluate the integrals:

(a) (*6 pts*) $\displaystyle\int_0^3 x\, e^{-x}\, dx =$

(b) (*6 pts*) $\displaystyle\int \frac{dx}{x(1+x)} =$

5. Evaluate the indefinite integrals:

(a) (*6 pts*) $\displaystyle\int \cos^2\theta \sin^2\theta \, d\theta =$

(b) (*6 pts*) $\displaystyle\int \frac{dz}{\sqrt{1+z^2}} =$

6. Determine whether the following series converge or diverge. Explain your reasoning and identify any test used.

(a) (*6 pts*) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

(b) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10+\sqrt{n}}$

7. **(a)** (*8 pts*) Find the Maclaurin series by any convenient method:

$$f(x) = x^2 e^{-x^2}$$

(b) (*8 pts*) Use the result in **(a)** to compute the following antiderivative as a power series:

$$\int x^2 e^{-x^2}\, dx =$$

8. **(a)** (*6 pts*) Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge? Explain your reasoning and identify any test used.

(b) (*5 pts*) Compute and simplify S_3 for the series in part **(a)**.

9. (*6 pts*) Make a careful and reasonably-large sketch of the cardiod $r = 1 - \sin\theta$. (*Label the axes and give dimensions/values along the axes.*)

10. Consider the parametric curve $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$.

(a) (*6 pts*) Find the equation of the tangent line at $t = 1$.

(b) (*6 pts*) Compute the second derivative $\dfrac{d^2y}{dx^2}$.

11. (*8 pts*) Find the radius and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n^2}$$

12. (*8 pts*) Find the general solution to the differential equation $y' = \ln x + \tan x$.

Extra Credit. (*3 pts*) First, explain in terms of familiar Maclaurin series, why $\dfrac{1}{e} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Second, how accurate is $S_9 = \displaystyle\sum_{n=0}^{9} \frac{(-1)^n}{n!}$ as an approximation to $1/e$? Write your bound on the error in the box below.

$$|R_9| = \left| S_9 - \frac{1}{e} \right| \le \boxed{}$$

You may find the following **trigonometric formulas** useful. However, there are other trig. formulas, not listed here, which you should have in memory, or which can be derived from these.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin(ax)\sin(bx) &= \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x) \\
\sin(ax)\cos(bx) &= \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x) \\
\cos(ax)\cos(bx) &= \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)
\end{aligned}$$

EXTRA SPACE FOR ANSWERS

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0,$ then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty,$ the series diverges.	
	If $\rho = 1,$ the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n,$ let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n.$
	If $\rho > 1$ or $\rho = \infty,$ the series diverges.	
	If $\rho = 1,$ the test is inconclusive.	