Math 252 Exam 3 Name: Key

11/18/2020 Instructor: Rhodes

Show all your work.

1. (20 pts.) A function $f(x)$ is defined by the series

$$f(x) = \sum_{n=0}^{\infty} \frac{5^n}{(n+1)2^n}(x-3)^n.$$

(a) (5 pts.) Determine the radius of convergence of the series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{5^{n+1}}{(n+3)2^{n+1}}(x-3)^{n+1}}{\frac{5^n}{(n+2)2^n}(x-3)^n}\right| = \frac{5}{2}\frac{n+2}{n+3}|x-3| \xrightarrow[\text{as } n\to\infty]{} \frac{5}{2}|x-3|$$

So the series converges if $\frac{5}{2}|x-3|<1$ or $|x-3|<\frac{2}{5}$

The radius of convergence is $\frac{2}{5}$

(b) (6 pts.) Determine the interval of convergence of the series.

If $x=3+\frac{2}{5}$ $\quad f(x) = \sum_{n=0}^{\infty}\frac{5^n}{(n+1)2^n}\left(\frac{2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$ harmonic series diverges

If $x=3-\frac{2}{5}$ $\quad f(x) = \sum_{n=0}^{\infty}\frac{5^n}{(n+1)2^n}\left(-\frac{2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\cdots$ alternating harmonic converges

The interval of convergence is $[2.6, 3.4)$

(c) (5 pts.) Give a series for $\int f(x)\,dx$.

$$C + \sum_{n=0}^{\infty}\frac{5^n}{(n+1)^2 2^n}(x-3)^{n+1}$$

(d) (4 pts.) What is $f^{(9)}(3)$?

$c_n = \frac{f^{(9)}(3)}{9!}$ so $f^{(9)}(3) = (9!)\left(\frac{5^9}{10\cdot 2^9}\right)$

2. (14 pts.) Using the definition, find the Taylor series for the function $f(x) = \cos x$ at $a = \frac{\pi}{4}$. Write out at least the first 5 terms.

$f(x) = \cos x$ $\quad f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(x) = -\sin x$ $\quad f'(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$f''(x) = -\cos x$ $\quad f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$f'''(x) = \sin x$ $\quad f'''(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f^{(4)}(x) = \cos x$ $\quad f^{(4)}(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

etc. $\quad$ etc

$$f(x) = \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{\sqrt{2}\cdot 2}(x-\frac{\pi}{4})^2 + \frac{1}{\sqrt{2}\cdot 3!}(x-\frac{\pi}{4})^3 + \frac{1}{\sqrt{2}\cdot 4!}(x-\frac{\pi}{4})^4 + \ldots$$

3. (14 pts. – 7 pts. each) Working from power series you already know, calculate at least the first 3 non-zero terms power series for the following.

(a) $\frac{1}{4+x}$ $= \frac{1}{4(1+\frac{x}{4})} = \frac{1}{4} \frac{1}{1-(-\frac{x}{4})} = \frac{1}{4}\left(1 + \frac{-x}{4} + (\frac{-x}{4})^2 + (\frac{-x}{4})^3 + \ldots\right.$

$$= \frac{1}{4} - \frac{x}{4^2} + \frac{x^2}{4^3} - \frac{x^3}{4^4} + \ldots$$

(b) $\frac{\sin x - x}{x^2}$ $= \frac{(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) - \cancel{x}}{x^2}$

$$= -\frac{x}{3!} + \frac{x^3}{5!} - \frac{x^5}{7!} + \frac{x^7}{9!} - \ldots$$

4. (24 pts. – 6 pts. each) Determine whether the following series converge or diverge. For each, *state what test you use*, and write enough to indicate that you checked all conditions necessary to apply the test.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

This is alternating $\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$, $\frac{1}{\ln n} \to 0$ as $n \to \infty$

So it converges.

(b) $\sum_{n=1}^{\infty} \frac{2n^3+7n+1}{n^6-5n}$

Limit comparison to $\frac{1}{n^3}$

$$\frac{\frac{2n^3+7n+1}{n^6-5n}}{\frac{1}{n^3}} = \frac{2n^6+7n^4+n^3}{n^6-5n} \to 2 \text{ as } n \to \infty$$

Since $\sum \frac{1}{n^3}$ converges (p-series, $p=3$)

this series converges.

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+2}{5n^3+n^2}$

$\frac{n^3+2}{5n^3+n^2} \to \frac{1}{5}$ so $a_n \not\to 0$ as $n \to \infty$

so by nth term test, the series diverges

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ for $n>1$ But $\sum \frac{1}{n^{1/2}}$ diverges (p-series $p=\frac{1}{2}$)

so the series diverges by the comparison test

5. (12 pts. – 6 pts. each) For the series

$$\sum_{n=0}^{\infty} \frac{2}{(2n+5)(2n+7)} = \sum_{n=0}^{\infty} \left(\frac{1}{2n+5} - \frac{1}{2n+7} \right),$$

(a) Find a formula for the nth partial sum, s_n.

$$s_n = \left(\frac{1}{5} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \cdots + \left(\cancel{\frac{1}{2n+5}} - \frac{1}{2n+7}\right)$$

$$= \frac{1}{5} - \frac{1}{2n+7}$$

(b) Using s_n, determine whether the series converges, and if it does, determine its limit.

$$s_n = \frac{1}{5} - \frac{1}{2n+7} \to \frac{1}{5} \quad \text{as } n \to \infty$$

So the series converges to $\frac{1}{5}$.

6. (14 pts. – 7 pts. each)

(a) Use the integral test to show the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges.

$a_n = f(n)$ for $f(x) = \frac{1}{x^4}$

Since $\int_1^\infty f(x)\,dx = \int_1^\infty x^{-4}\,dx = \lim_{b\to\infty} \int_1^b x^{-4}\,dx = \lim_{b\to\infty} \left(\frac{x^{-3}}{-3} \Big|_1^b \right)$

$= \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3} \right) = \frac{1}{3} < \infty$, the series converges

(b) The remainder estimate for the integral test says that if the series converges to s, then the difference $R_n = s - s_n$ between the series and its nth partial sum satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx.$$

Use this to determine how many terms of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ need to be added up to estimate s to within 10^{-10}

$$s - s_n \le \int_n^\infty x^{-4}\,dx = \lim_{b\to\infty} \left(\frac{x^{-3}}{-3} \Big|_n^b \right) = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3n^3} \right)$$

$$= \frac{1}{3n^3}.$$

To make $s - s_n < 10^{-10}$ we make $\frac{1}{3n^3} < 10^{-10}$

i.e. $10^{10} < 3n^3$ or $n > \sqrt[3]{\frac{10^{10}}{3}}$