Assignment #3

Due Monday, 10 February 2020, at the start of class

Please read Chapter 8 of the textbook. Also review the slides *Finite-dimensional spectral theory I* at bueler.github.io/M617S20/slides1.pdf

One exercise below is identified with your initials. Please LaTeX this problem and send both the `.tex` and `.pdf` to me at `elbueler@alaska.edu` by the due date. See the course website for a LaTeX template.

DO THE FOLLOWING EXERCISES from the textbook (Muscat, *Functional Analysis*, 2014):

- #1 in Exercises 8.10, pages 125–126.
- #3 in Exercises 8.10, page 126. *There are several things to prove. Operator L is defined on page 118.*
- #4 in Exercises 8.10, page 126.
- #5 in Exercises 8.10, page 126. *Note ℓ_1^1 is defined in #4.*
- #10 in Exercises 8.10, page 126. $\longleftarrow$ **DD**
- #12 in Exercises 8.10, page 126. $\longleftarrow$ **WV**
- #19 in Exercises 8.10, page 127. $\longleftarrow$ **OS**
- #4 in Exercises 8.14, page 129. *Prove both claims in the last sentence.*

ALSO DO THE FOLLOWING EXERCISE:

- **Exercise B.** Consider the inner product space $V = L^2([0,\infty), e^{-x})$, that is, the set of a.e. equivalence classes of measurable functions $f : [0,\infty) \to \mathbb{C}$ such that $\int_0^\infty |f(x)|^2 e^{-x}\,dx < \infty$, and with inner product and norm

$$\langle f, g\rangle = \int_0^\infty \overline{f(x)} g(x)\, e^{-x}\,dx, \qquad \|f\| = \left(\int_0^\infty |f(x)|^2 e^{-x}\,dx\right)^{1/2}.$$

(*In fact V is a Hilbert space (Chapter 10), but this exercise does not use completeness.*)

(a) Show that monomials x^n, for $n = 0, 1, 2, \dots$, are in V.

(b) Show that the set of monomials $\{1, x, x^2, x^3, \dots\}$ is linearly-independent in V. (*Hint.* Consider $\lim_{x\to\infty} x^{n+1}/|p(x)|$ if $p(x)$ is a nonzero polynomial of degree at most n. This shows x^{n+1} is not in the span of $\{1, x, \dots, x^n\}$.)

(c) Apply the Gram-Schmidt process in V to the set $S = \{1, x, x^2, x^3\}$ to compute polynomials $\hat L_0(x), \hat L_1(x), \hat L_2(x), \hat L_3(x)$ which are orthonormal (ON) and span the space of degree 3 polynomials.

(d) Renormalize the polynomials $\hat L_j(x)$ to a new set $L_0(x), L_1(x), L_2(x), L_3(x)$ such that $L_j(0) = 1$. These are the first four *Laguerre* polynomials; check your work at en.wikipedia.org/wiki/Laguerre_polynomials

EC. (Extra Credit) Write a short MATLAB program which approximates and plots the polynomials $L_i(x)$. (*My code is six short lines.*)