

# Matrix Norm Essentials

- Matrix norms have vector norm **properties**:

- $\|A\| \geq 0$  and  $\|A\| = 0 \implies A = 0$
- $\|A + B\| \leq \|A\| + \|B\|$
- $\|\alpha A\| = |\alpha| \|A\|$

- Induced norms, and the Frobenius norm, have additional **multiplicative** property

- $\|AB\| \leq \|A\| \|B\|$

- Really only **four** norms to know:

$$\|\cdot\|_1, \quad \|\cdot\|_2, \quad \|\cdot\|_\infty, \quad \|\cdot\|_{\text{Frob}}$$

- 3 have **easy-to-compute formulas**:  $1, \infty, \text{Frob}$
- 3 are **induced** from vector norms:  $1, 2, \infty$

- Induced norms have  $\rho(A) \leq \|A\|$ .

- But **expect**  $\rho(A) < \|A\|$  in general, and sometimes  $\rho(A) \ll \|A\|$ .

- $\|\cdot\|_2$  norm best for **Euclidean ideas** and **hermitian/normal matrices**. Reasons:

- $\|QA\|_2 = \|A\|_2$  if  $Q$  is unitary ( $Q^*Q = I$ ).
- $\sigma_1(A) = \|A\|_2$ .
- If  $A^* = A$  then  $\rho(A) = \|A\|_2$ .

- **Iteration**  $v, Av, A^2v, \dots$  converges if and only if

$$\rho(A) < 1.$$

- Thus **if**  $\|A\| < 1$  **then** convergence. Not conversely!