Worksheet: Computing condition numbers.

The goal of this worksheet is to demystify condition numbers. Use the Formulas below, which are from Lecture 12, and your knowledge of norms (Lecture 2), to do the Exercises at the bottom and on the next page. Refer to the text as needed.

Formulas. A *problem* is a function $f : X \to Y$, where X and Y are normed vector spaces. The *Jacobian matrix* of a problem f is its first derivative. Specifically, if $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ then the ith component of the output is $f_i(x) = f_i(x_1, \dots, x_n)$ and the Jacobian is an $m \times n$ matrix:

$$J = J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{1}$$

Condition numbers measure the sensitivity of problems. They compare output changes to input changes when the input change is small. The *absolute condition number* of a problem f is defined as $\hat\kappa = \sup_{\delta x} \|\delta f\|/\|\delta x\|$, but when f has a derivative one may compute $\hat\kappa$ as

$$\hat\kappa = \hat\kappa_f(x) = \|J(x)\|. \tag{2}$$

The *relative condition number* has a more relative definition, $\kappa = \sup_{\delta x}(\|\delta f\|/\|f(x)\|) \,/\, (\|\delta x\|/\|x\|)$, but when f has a derivative one may compute

$$\kappa = \kappa_f(x) = \frac{\|J(x)\|}{\|f(x)\|/\|x\|}. \tag{3}$$

Exercises. For each of the 4 problems below use formulas (1), (2), and (3) to compute J, $\hat\kappa$, and κ. When you have a choice of norms, choose the most convenient one.

1. $f : (0, \infty) \to \mathbb{R}$ has formula $f(x) = 1/x$.

2. $f : \mathbb{R}^2 \to \mathbb{R}$ has formula $f(x) = x_1 + x_2$.

3. $f : \mathbb{R}^n \to \mathbb{R}$ has formula $f(x) = x_1^2 + x_2^2 + \cdots + x_n^2$.

4. $f : \mathbb{R}^n \to \mathbb{R}^m$ has formula $f(x) = Ax$ where A is a fixed $m \times n$ matrix.