Assignment #8

Due Monday 8 November, 2021 at the start of class

Please read Lectures 15, 16, 17, and 20 in the textbook *Numerical Linear Algebra* by Trefethen and Bau. Then do the following exercises.

P21. *The major goal of this exercise is to show that the usual matrix-vector product algorithm is backward-stable when we regard the* matrix *as the input; see part* **(c)**. *For simplicity I'll assume all entries are real numbers. Always assume axioms (13.5) and (13.7) hold.*

(a) Let us start in one dimension. Fix $x \in \mathbb{R}^1$. Show that if the problem is scalar multiplication, $f(a) = ax$ for $a \in \mathbb{R}^1$, then the obvious algorithm $\tilde{f}(a) = \mathrm{fl}(a) \otimes \mathrm{fl}(x)$ is backward-stable.

(b) Fix $x \in \mathbb{R}^2$. Show that if $a \in \mathbb{R}^2$ then the obvious algorithm $\tilde{f}(a) = \mathrm{fl}(a_1) \otimes \mathrm{fl}(x_1) \oplus \mathrm{fl}(a_2) \otimes \mathrm{fl}(x_2)$ for the inner product problem $f(a) = a^\top x$ is backward-stable. (*Hint. Both a and x are column vectors. You must choose a vector norm to finish the proof.*)

A proof by induction extending part **(b)** *shows that the obvious inner product algorithm is backward-stable in any dimension; see Example 15.1. From now on you can assume it is true.*

(c) Fix $x \in \mathbb{R}^n$. Show that if $A \in \mathbb{R}^{m \times n}$ then the obvious algorithm $\tilde{f}(A)$ for the matrix-vector product problem $f(A) = Ax$ is backward-stable. (*Hints. Start by expressing Ax using inner products. Do not get your hands dirty with scalar entries of A or x. You must pick a vector norm and an induced norm.*)

(d) Fix $A \in \mathbb{R}^{m \times n}$. Explain in at most 8 sentences why the obvious algorithm $\tilde{f}(x)$ for the matrix-vector product problem $f(x) = Ax$ is generally ***not*** backward-stable. However, this result depends on dimension. In fact, for what m, n do we already know that this $\tilde{f}(x)$ ***is*** backward-stable? (*Hints. The problem and algorithm are actually "the same" as in part* **(c)**, *but now the input is x. Consider what we know for inner products.*)

(e) Is the standard algorithm for matrix-matrix multiplication backward-stable? Regard the problem with maximum flexibility, that is, as treating *both* factors as inputs: $F(A, B) = AB$ for $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$. Explain in at most 8 sentences. (*Hints. The answer depends on the dimensions. You must be precise, and consider different cases for the dimensions, in order to get full credit.*)

Exercise 15.2.

Exercise 16.2.

Exercise 17.2.