3 kinds of integrals
and 2 kinds of anti-derivatives

Ed Bueler

Math 422 Intro Complex Analysis

15 March 2016

purpose

- these slides are easier to read than the book's text
- they will help you do Assignment #7
- but you still need to read the book!
  - read sections I.6, IV.1, IV.2, and IV.3
- my goal:

  if you spend time on these slides, and on the above sections of the book, then you will find Assignment #7 doable. . . and perhaps easy

3 kinds of integrals over curves

- suppose γ is a smooth curve in the plane $\mathbb{R}^2 = \mathbb{C}$
  - it can be parameterized $\gamma(t) = (x(t), y(t))$ for $a \le t \le b$
- suppose $P(x,y)$, $Q(x,y)$, and $f(z)$ are continuous functions
- here are 3 kinds of integrals over γ:

(A) $$\int_\gamma P\,dx + Q\,dy$$

(B) $$\int_\gamma f(z)\,dz$$

(C) $$\int_\gamma f(z)\,|dz|$$

general advice on these integrals

- it is easy to get confused with these closely-related integral concepts
- the book calls both B and C "complex line integrals", and it gives no names to distinguish the two
- A is a Chapter III topic . . . we already have practice
- to keep track of the differences you will need *both*
  - to pay attention to the notation, and
  - to trust the notation to tell you what to do

concrete examples of all three

- let's compute A, B, C for the same specific curve:

$$\gamma(t) = (x(t), y(t)) = (t, 1-t), \qquad 0 \le t \le 2$$

- assume $P(x,y) = xy$ and $Q(x,y) = \sqrt{x} + y$:

A
$$\begin{aligned}
\int_\gamma P\,dx + Q\,dy &= \int_a^b P\frac{dx}{dt}\,dt + Q\frac{dy}{dt}\,dt \\
&= \int_0^2 t(1-t)(1) + (\sqrt{t} + 1 - t)(-1)\,dt \\
&= \int_0^2 2t - t^2 - \sqrt{t} - 1\,dt \\
&= 2^2 - \frac{2^3}{3} - \frac{2}{3}2^{3/2} - 2 = -\frac{2}{3}(1 + 2\sqrt{2})
\end{aligned}$$

- this γ is not a closed curve, so Green's theorem is not an option
  - ...but it was not needed either

concrete examples 2

- continuing with same path $\gamma(t)$, now assume $f(z) = z^2$:

$$\text{B} \qquad \int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\frac{dz}{dt}\,dt$$
$$= \int_a^b f(x(t) + iy(t))(x'(t) + iy'(t))\,dt$$
$$= \int_0^2 (t + i(1-t))^2\,(1-i)\,dt$$
$$= (1-i)\left(\int_0^2 2t - 1\,dt + 2i\int_0^2 t - t^2\,dt\right)$$
$$= (1-i)\left(2 - 2i\frac{2}{3}\right) = \frac{2}{3} - i\frac{10}{3}$$

- we will see that if $f(z)$ is analytic, as in this case, then B-type integrals are path-independent
  - *Cauchy' Theorem*: if γ is closed and if $f(z)$ is analytic, then $\oint_\gamma f(z)\,dz = 0$
  - in above example γ is not closed, but answer would be the same for another curve with same starting and ending points

concrete examples 3

- continuing with same curve $\gamma(t)$, and same $f(z)$:

$$\text{C} \qquad \int_\gamma f(z)\,|dz| = \int_a^b f(x(t)+iy(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$$
$$= \int_0^2 (t+i(1-t))^2\,\sqrt{1+1}\,dt$$
$$= \sqrt{2}\left(\int_0^2 2t-1\,dt + 2i\int_0^2 t-t^2\,dt\right)$$
$$= \sqrt{2}\left(2-2i\frac{2}{3}\right) = 2\sqrt{2} - i\frac{4\sqrt{2}}{3}$$

- note $|dz| = ds$ is "element of arclength"
- such C-type integrals are (essentially) never path-independent *even when $f(z)$ is analytic*
  - only exception is when $f(z)=0$

summary: general forms for parameterized curves

- if $\gamma(t) = (x(t), y(t))$, $a \le t \le b$, is smooth or piecewise smooth, and if $P(x,y)$, $Q(x,y)$, $f(z)$ are continuous, then

(A)
$$\begin{aligned}\int_\gamma P\,dx + Q\,dy &= \int_a^b P\frac{dx}{dt}\,dt + Q\frac{dy}{dt}\,dt \\ &= \int_a^b P\,(x(t),y(t))\,x'(t) + Q\,(x(t),y(t))\,y'(t)\,dt\end{aligned}$$

(B)
$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(z(t))\frac{dz}{dt}\,dt \\ &= \int_a^b f\,(x(t)+iy(t))\,\big(x'(t)+iy'(t)\big)\;dt\end{aligned}$$

(C)
$$\begin{aligned}\int_\gamma f(z)\,|dz| &= \int_a^b f(z(t))\left|\frac{dz}{dt}\right|\,dt \\ &= \int_a^b f\,(x(t)+iy(t))\,\sqrt{x'(t)^2+y'(t)^2}\,dt\end{aligned}$$

- key idea: if you use the notation carefully and consistently then it will tell you how to expand until you are integrating a function of t

comments, and quicker notation

- remaining slides focus on B- and C-type integrals, which are the subject of IV.1 and IV.2
- when a curve γ is closed, tradition says to add a circle to the integral symbol
  - any path γ:
$$\int_\gamma f(z)\,dz$$
  - closed path γ:
$$\oint_\gamma f(z)\,dz$$
- writing the curve γ as "$z(t)$" for "$x(t) + iy(t)$", instead of "$(x(t), y(t))$", often makes computations quicker (next few slides)
  - especially for B-type integrals $\int_\gamma f(z)\,dz$
  - note: if γ is closed, and given by $z(t)$ on $a \le t \le b$, then
$$z(a) = z(b)$$

integrals of 1 on closed paths

- easy case of (B) over any closed curve:

$$\oint_\gamma 1\,dz = \int_a^b z'(t)\,dt = z(b) - z(a) = 0$$

- easy case of (C) over any closed curve:

$$\oint_\gamma 1\,|dz| = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt = (\text{length of } \gamma)$$

  - this integral just computes the arclength, from calculus:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

easy exponential integrals on closed paths

- easy case of (B) over the unit circle γ centered at the origin: $z(t) = e^{it}$, $0 \leq t \leq 2\pi$:

$$\oint_\gamma e^z\,dz = \int_0^{2\pi} e^{z(t)} z'(t)\,dt \overset{*}{=} e^{z(2\pi)} - e^{z(0)} = 0$$

- key step $*$ is the Fundamental Theorem of Calculus (see below)
- plus the chain rule:

$$\frac{d}{dz} e^z = e^z \qquad \text{so} \qquad \frac{d}{dt}\left(e^{z(t)}\right) = e^{z(t)} z'(t)$$

- the value comes out zero for any closed curve; it did not actually matter that γ was a circle

easy exponential integrals on closed paths 2

- easy case of Ⓒ over same circle; note $z(t) = e^{it}$ so $x(t) = \cos t$, $y(t) = \sin t$:

$$\begin{aligned}
\oint_\gamma e^z\,|dz| &= \int_0^{2\pi} e^{(e^{it})}\sqrt{x'(t)^2 + y'(t)^2}\,dt \\
&= \int_0^{2\pi} e^{(e^{it})}\sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} e^{(e^{it})}\,dt \\
&\overset{\dagger}{=} 2\pi e^0 = 2\pi
\end{aligned}$$

- key step † is Mean Value Property (section III.4) with $z_0 = 0$ and $r = 1$ and $f(z) = e^z$:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)\,d\theta$$

- it was important that γ is a circle; on other closed curves the result would be different

the complex Fundamental Theorem of Calculus

- generally on any path γ of the form $z(t)$ for $a \leq t \leq b$, if $F'(z) = f(z)$ then

$$\int_\gamma f(z)\,dz = F(z(b)) - F(z(a))$$

- *Proof*. By chain rule,

$$\frac{d}{dt}\left[F(z(t))\right] = F'(z(t))\,z'(t) = f(z(t))\,z'(t).$$

Thus

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt}\left[F(z(t))\right]\,dt$$
$$= F(z(b)) - F(z(a)) \quad \square$$

B-type integrals over closed paths

- by FTC on last slide, B-type integrals over closed paths come out zero under a condition
- namely: *if* there is $F(z)$ for which $F'(z) = f(z)$ *then*

$$\oint_\gamma f(z)\,dz = F(z(b)) - F(z(a)) = 0$$

because $z(b) = z(a)$ if γ is closed

B-type integrals over closed paths 2

- there is another justification of why these integrals are zero, under a different condition
- namely: assume γ is the boundary of a domain D
- also write $f(z) = u(x, y) + iv(x, y)$ and $\gamma(t) = (x(t), y(t))$
- if $f(z)$ is analytic in D, and on $\gamma = \partial D$, then

$$\begin{aligned}
\oint_\gamma f(z)\,dz &= \int_a^b (u(x,y) + iv(x,y))(x' + iy')\,dt \\
&= \left[\oint_\gamma u\,dx - v\,dy\right] + i\left[\oint_\gamma v\,dx + u\,dy\right] \\
&\overset{\text{Green's Thm}}{=} \left[\iint_D \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\,dx\,dy\right] + i\left[\iint_D \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dx\,dy\right] \\
&\overset{\text{Cauchy-Riemann}}{=} 0
\end{aligned}$$

  - this Cauchy's Theorem as stated in section IV.3
  - anti-derivative $F(z)$ is never mentioned!

connection to the Mean Value Property (MVP)

- assume $f(z)$ is analytic on some domain which includes the disk of radius $R > 0$ around z_0
- if curve γ is a circle, i.e. $z(t) = z_0 + r\,e^{it}$ with $r < R$, then by MVP

$$\oint_\gamma f(z)\,|dz| = \int_0^{2\pi} f(z_0 + r\,e^{it})\,r\,dt = 2\pi r\,f(z_0)$$

- for example, on slide 12 the C-type integral was done by MVP
- by contrast, the integral with "dz" instead of "$|dz|$" is zero (by chain rule or Cauchy's Theorem):

$$\begin{aligned}\oint_\gamma f(z)\,dz &= \int_0^{2\pi} f(z_0 + r\,e^{it})\,ir\,e^{it}\,dt \\ &= f(z_0 + r\,e^{i0}) - f(z_0 + r\,e^{i2\pi}) = 0\end{aligned}$$

extra: the integrals at the start of section IV.1

- we seem to see yet another form of integral on page 102 of the book, namely

$$\int h(z)\,dz = \int h(z)\,dx + i\int h(z)\,dy$$

- …a B-type integral written with $dz = dx + idy$ expanded
- nothing new or special! …merely allows the author to recall definition of integral over curve:

$$\begin{aligned}\int h(z)\,dz &= \lim_{n\to\infty} \sum_{n \text{ parts}} h(z_j)(z_{j+1} - z_j) \\ &= \lim_{n\to\infty} \sum_{n \text{ parts}} h(z_j)(x_{j+1} - x_j) + i \lim_{n\to\infty} \sum_{n \text{ parts}} h(z_j)(y_{j+1} - y_j)\end{aligned}$$

with the curve γ broken-up into n parts as shown on page 103

2 kinds of anti-derivatives

- anti-derivatives are important because of the FTC
- there are two kinds of anti-derivatives in $\mathbb{R}^2 = \mathbb{C}$:

  i) a potential $h(x, y)$ for which $dh = P\,dx + Q\,dy$

  ii) a primitive $F(z)$ for which $F'(z) = f(z)$

- $h(x, y)$ is real-valued, while $F(z)$ is complex-valued
- suppose D is a star-shaped domain, so D is open, connected, and has no holes (simply-connected)
  - a potential exists if $P(x, y)$ and $Q(x, y)$ satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, so that $P\,dx + Q\,dy$ is closed
  - a primitive exists if $f(z)$ is analytic, so Cauchy-Riemann equations apply

easy anti-derivative example: type (i)

- for a type (i) example, consider

$$P\,dx + Q\,dy = e^x \cos y\,dx - e^x \sin y\,dy$$

- it is closed because $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^x \sin y + e^x \sin y = 0$
- we find a potential $h(x, y)$ by finding anti-derivatives one variable at a time, first using the idea that $\frac{\partial h}{\partial x} = P$ and then using $\frac{\partial h}{\partial y} = Q$:

$$\begin{aligned} h(x,y) &= e^x \cos y + r(y) \\ -e^x \sin y &= \frac{\partial h}{\partial y} = -e^x \sin y + r'(y) \\ &\quad (\iff r'(y) = 0 \iff r(y) = C) \\ h(x,y) &= e^x \cos y + C \end{aligned}$$

where $C \in \mathbb{R}$

easy anti-derivative example: type ii

- for a type ii example, consider

$$f(z) = e^z$$

- then

$$F(z) = e^z + C$$

  is a primitive because $(e^z)' = e^z$; here $C \in \mathbb{C}$
- yes, it can be this easy . . . for these underlying reasons:
  - we already have a mental supply of one-variable anti-derivatives
  - the complex number $z = x + iy$ is the "right way" to combine two real variables into one symbol, at least if you want to do algebraic jobs like taking by-hand derivatives or computing by-hand anti-derivatives

complex logarithms: the key facts you need

- this final slide is about section I.6, but is useful when doing exercises from IV.1 and IV.2
- *claim*. all computations with the complex logarithm can be done by these formulae:

  - $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z \qquad \text{on } \mathbb{C} \setminus \{0\}$
  - $\log z = \{\operatorname{Log} z + 2\pi m i \;:\; m \text{ an integer}\}$
  - $e^{\log z} = z$
  - $\dfrac{d}{dz}(\operatorname{Log} z) = \dfrac{1}{z} \qquad \text{on } \mathbb{C} \setminus (-\infty, 0]$

  - the last fact is from the Example on page 108
- understanding these facts requires that you *already* understand these functions, so review appropriately:
  - $\log x$ for $x > 0$ ... just the ordinary natural logarithm (= $\ln x$)
  - $\operatorname{Arg} z$ on $\mathbb{C} \setminus \{0\}$
  - e^z on $\mathbb{C}$