Review Topics for in-class Midterm Exam I on *Friday 19 February, 2015*

Midterm Exam I will cover these sections in Gamelin, *Complex Analysis*:

I.1, I.2, I.3, I.4, I.5, II.1, II.2, II.3

The problems will be of these types: state definitions, state theorems, describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems.

In everything below, $z = x + iy$ is a complex number and $f(z) = u(x, y) + iv(x, y)$ is a complex-valued function.

Definitions. Know how to define:

- zw (= complex multiplication)
- $|z|$ and $\bar{z}$
- $e^{i\theta}$ and e^z
- $\arg z$ and $\operatorname{Arg} z$
- principal branch of square root (p. 17)
- $\{s_n\}$ converges to s (*See definition pages 33–34. Know both the informal and the formal definition. Also, know that "$s_n \to s$" and "$\lim_{n\to\infty} s_n = s$" denote the same concept.*)
- $\lim_{z\to z_0} f(z) = L$
- $f(z)$ is continuous at z_0
- open set
- domain
- convex set and set is star-shaped with respect to z_0
- $f(z)$ is differentiable at z_0, and $f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$
- $f(z)$ is analytic on an open set
- Cauchy-Riemann equations (equations (3.3) on p. 47)

Theorems. Understand the theorems and be able to use as facts. Be able to prove if it says so.

- $|z + w| \le |z| + |w|$, the triangle inequality
- the Fundamental Theorem of Algebra (p. 4)
- the three-part Theorem on page 34 [**Be able to prove the sum case.**]
- the in-between ("squeeze") Theorem (p. 35)
- Theorem (p. 35): A bounded monotone sequence of real numbers converges.
- Theorem (p. 36): $\{s_n\}$ converges if and only if $\{\operatorname{Re} s_n\}$ and $\{\operatorname{Im} s_n\}$ converge.
- Lemma (p. 36): $\lim_{z\to z_0} f(z) = L$ if and only if $\lim_{n\to\infty} f(s_n) = L$ for every sequence s_n which converges to z_0 (and is in the domain of $f(z)$).
- the three-part Theorem on page 37
- Theorem (p. 38): If $h(x, y)$ has continuous first derivatives and they are zero ($\nabla h = 0$) on a domain then h is constant.

- Theorem (p. 43): If $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.
- derivative rules (2.5)–(2.9) **[Be able to prove ANY OF THESE.]**
- Chain Rule theorem (p. 44)
- Theorem (p. 47): $f(z) = u+iv$ is analytic if and only if Cauchy-Riemann equations apply to u and v **[Be able to prove that if f is analytic then Cauchy-Riemann equations.]**
- Theorem (p. 49): If $f(z)$ is analytic and $f'(z) = 0$ on a domain then $f(z)$ is constant.

Calculations. Know how to do basic examples. **Be able to prove** (= show why the formula is true in general, starting from definitions) **any of these**.

- $z\bar{z} = |z|^2$
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$
- $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$
- how $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$ gives trigonometric identities
- how $\left(e^{i\theta}\right)^n = e^{in\theta}$ gives trigonometric identities (de Moivre)
- polar formulas for nth powers and nth roots (p. 8)
- polar formulas for nth roots of unity (p. 9)
- sum of geometric series (p. 10): $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$
- formulas for principal and second branches of the square root (p. 17)
- formulas for all branches of nth root
- $e^{z+w} = e^z e^w$ and $1/e^z = e^{-z}$
- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $0 < p < \infty$
- $\lim_{n\to\infty} z^n = 0$ if $|z| < 1$
- limits like 1(a)(b) on page 39, using the definition of the limit
- $n \to \infty$ limits of rational functions like example on page 35, using $\lim_{n\to\infty} \frac{1}{n^p} = 0$ etc.
- derivative of $f(z) = z^m$ is $f'(z) = mz^{m-1}$
- derivatives of polynomials and rational functions
- apply Cauchy-Riemann equations to show $f(z) = u + iv$ is analytic or not analytic
- facts like Exercises 3,4,5 on page 50

Sketchables/Describeables. Know how to show, or describe in clear words, some basic examples. *Regarding sketches: Make it adequately big! Give at least one indication of scale on each axis.*

- plot complex numbers from cartesian or polar form
- plot all nth roots of a complex number
- plot stereographic projection, and inverse stereographic projection, of simple sets **[do not worry about formulas of stereographic projection]**
- plot z-plane and w-plane of branches of nth roots (p. 17)
- plot images of circles and rays under z^n
- plot images of horizontal and vertical lines under e^z
- sketch examples of sequence limits (p. 34)
- open sets, domains, convex sets, star-shaped sets (pp. 37–39)