A closed ball in a big vector space is not compact

The title should probably be "in at least one infinite-dimensional vector space, a closed ball of radius 2 centered at the origin is not compact," but that is an awkward title. It is not too hard to extend the idea here, and one can show that in a infinite-dimensional Hilbert space[1] every non-empty closed ball is not compact.

First, I claim a general fact about metric spaces, that a collection of isolated singletons is closed, even if there are infinitely many such singletons. To understand it, one should sketch the balls used in the proof.

Lemma. *Suppose that (X, d) is a metric space. Suppose $W \subseteq X$ has the property that every element of W is isolated away from every other element by a distance $\delta > 0$, in the precise sense that, for this $\delta > 0$, $x \in W$ implies $B_\delta(x) \cap W = \{x\}$. Then W is closed.*

Proof. We show $\overline{W} \subseteq W$ so that $W = \overline{W}$ and thus W is closed. Let $y \in \overline{W}$. Then there is $x \in B_\delta(y) \cap W$, by definition 6.7. But then $y \in B_\delta(x)$.

If $y \notin W$ then $x \neq y$ so $d(x, y) > 0$. Noting $0 < d(x, y) < \delta$, let

$$\epsilon = \min\left\{\frac{1}{2}d(x,y), \delta - d(x,y)\right\} > 0.$$

Let $z \in B_\epsilon(y)$. Then

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \epsilon \le d(x,y) + \delta - d(x,y) = \delta$$

so $z \in B_\delta(x)$. We have proven that $B_\epsilon(y) \subseteq B_\delta(x)$. On the other hand, $x \notin B_\epsilon(y)$ because otherwise $d(x, y) < \epsilon \le \frac{1}{2}d(x, y)$, a contradiction. So now $B_\epsilon(y) \cap W = \emptyset$ because otherwise $B_\delta(x) \cap W \supseteq B_\epsilon(y) \cap W$ contains a point other than x. This contradicts $y \in \overline{W}$. So in fact $y \in W$ and thus W is closed. □

Now recall the vector space I described in class:

$$X = \{f : [0, 1] \to \mathbb{R} \,:\, f \text{ is continuous}\}.$$

(The standard notation is $X = C([0, 1])$.) We define both an inner product and a metric,

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx,$$

$$d(f,g) = \left(\int_0^1 (f(x) - g(x))^2\,dx\right)^{1/2} = \langle f - g, f - g\rangle^{1/2}.$$

One can find an infinite orthonormal set in X. To be concrete, for $k = 1, 2, \dots$ let

$$f_k(x) = \sqrt{2}\sin(k\pi x).$$

Trigonometric integrals—a good exercise in calculus—give $\langle f_j, f_k\rangle = 0$ if $j \neq k$, while $\langle f_k, f_k\rangle = 1$. Note that if $j \neq k$ then

$$d(f_j, f_k)^2 = \langle f_j - f_k, f_j - f_k\rangle = \langle f_j, f_j\rangle - 2\langle f_j, f_k\rangle + \langle f_k, f_k\rangle = 1 - 0 + 1 = 2.$$

[1]See en.wikipedia.org/wiki/Hilbert_space.

Let

$$W = \bigcup_{k=1}^{\infty} \{f_k\} = \{f_1, f_2, \ldots, f_k, \ldots\}.$$

If $u, v \in W$ are distinct then $d(u, v) = \sqrt{2}$. Apply the lemma with $\delta = \sqrt{2}$ to conclude W is closed. This fact removes the technical hang-up, experienced in lecture, in showing that there is a closed and bounded set that is not compact.

Proposition. *The closed radius 2 ball in X centered at the origin, namely*

$$C = \overline{B_2(0)} = \{f \in X \,:\, d(f, 0) \leq 2\},$$

is not compact.

Proof. Let $U_k = B_{1/2}(f_k)$, an open set containing f_k. Note that $U_j \cap U_k = \emptyset$ if $j \neq k$ because if $g \in U_j \cap U_k = B_{1/2}(f_j) \cap B_{1/2}(f_k)$ then

$$\sqrt{2} = d(f_j, f_k) \leq d(f_j, g) + d(g, f_k) < \frac{1}{2} + \frac{1}{2} = 1,$$

which is false. Now, with W defined above,

$$\mathcal{U} = \{X \setminus W\} \cup \{U_1, U_2, \ldots, U_k, \ldots\}$$

is an infinite open cover of C. In particular, if $h \in C \cap W$ then $h = f_k$ for some k and thus $h \in U_k$, while if $x \in C \setminus W$ then $x \in X \setminus W$. On the other hand, no proper subcover of $\mathcal{U}$ is a cover. This is because $\mathcal{U} \setminus \{U_k\}$ does not contain $f_k \in C$ and $\mathcal{U} \setminus \{X \setminus W\}$ does not contain $0 \in C$ (in particular). Thus C is not compact. $\square$