Author: Ed Bueler

Date: 1 September 2013

Exercise 6.66. The square root of 2 is irrational.

Proof. Suppose to the contrary that $\sqrt{2}$ is rational. Then there are integers a and b with no common factors such that

$$\sqrt{2} = \frac{a}{b}.$$

Squaring this equation we find that

$$2b^2 = a^2. \tag{1}$$

Hence 2 divides a^2 and therefore a^2 is even. If a were odd, then a^2 would also be odd, which it is not, so we conclude that a is even. So $a = 2k$ for some integer k. It follows from equation (1) that

$$\begin{aligned} 2b^2 &= (2k)^2 \\ &= 4k^2. \end{aligned}$$

Hence

$$b^2 = 2k^2.$$

Arguing as before we see that b^2, and thus also b itself, must be even. So 2 is a common factor of a and b, which is a contradiction. $\square$