

# Construction of steady state solutions for isothermal shallow ice sheets

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ABSTRACT. Exact solutions for ice sheet equations can and should play an important role in numerical model validation. In constructing a time-dependent exact solution to the thermocoupled shallow ice approximation, it was discovered that the existing analytical *steady* solutions are a poor basis [1]. This prompted a search for cleaner steady solutions to the isothermal shallow ice equation. In particular, we seek smooth solutions, at least between central peak and margin, which have the physically realistic property, for a grounded ice sheet, of margin-in-ablation-zone. The source of cleaner solutions is identified here as a flux function  $Q$  whose  $n$ th root is integrable, where  $n$  is the Glen exponent. Radially symmetric and one-horizontal dimension examples are identified.

## Introduction

The *steady, isothermal* cold shallow ice equation [7] is

$$(1) \quad a = \nabla \cdot \mathbf{Q}, \quad \mathbf{Q} = -\frac{\Gamma}{n+2} h^{n+2} |\nabla h|^{n-1} \nabla h, \quad h \geq 0$$

as an equation for surface elevation  $h = h(x, y)$  in a region in  $\mathbf{R}^2$ . Equation (1) is the special case of an ice sheet on a flat, rigid bed, so that surface elevation and thickness coincide. Here  $a = a(x, y)$  is the ice-equivalent accumulation/ablation function and  $\mathbf{Q} = \mathbf{Q}(x, y)$  is the vertically-integrated horizontal ice flux. The constants  $\Gamma > 0$  and  $n > 1$  represent material properties. The former depends strongly on the constant value of temperature.

Equation (1), for  $h$  given  $a$ , is second order in space and is elliptic as a PDE for  $h$ . More precisely, in regions where  $h > 0$ , (1) can be written as a quasilinear PDE for  $h$  by eliminating  $\mathbf{Q}$ , that is,  $a = -\nabla \cdot (\frac{\Gamma}{n+2} h^{n+2} |\nabla h|^{n-1} \nabla h)$ . (A PDE is *quasilinear* if it is linear in the highest derivative.) In regions where  $h|\nabla h| > 0$  the PDE is second-order in space and strictly-speaking elliptic. Generally, we say that the PDE is “degenerate elliptic.” Most essentially, (1) is a free boundary value problem with a one-sided constraint and more-or-less a “variational inequality” [2, 3].

## One horizontal dimension

Suppose  $h = h(x)$  and therefore  $\nabla(h(x)) = h'(x)$  and  $\nabla \cdot (Q(x)\hat{\mathbf{x}}) = Q'(x)$ . Let us also suppose that  $x = 0$  represents an ice-ridge and therefore  $Q(0) = 0$ . If there is

only one ridge then  $h'(x) \leq 0$  for  $x > 0$ , at least when  $h'$  is defined. We compute that  $Q(x) = \int_0^x a(\xi) d\xi$  and thus  $\frac{\Gamma}{n+2} h(x)^{n+2} (-h'(x))^n = \int_0^x a$ . Taking the  $(1/n)$ th power and treating as a separable ODE gives

$$(2) \quad h(x)^{2+2/n} = h_0^{2+2/n} - C_1 \int_0^x Q(\xi)^{1/n} d\xi = h_0^{2+2/n} - C_1 \int_0^x \left( \int_0^\xi a \right)^{1/n} d\xi, \quad x \geq 0,$$

where  $C_1 = (2 + 2/n)(n + 2)^{1/n} \Gamma^{-1/n}$ . Note that if  $a$  is even then  $Q$  is odd and  $h$  is even. These calculations are well-known [7].

Our goal here is to find convenient, exact solutions to (1), representing vaguely realistic physical situations, suited to the testing of numerical procedures. From (2), such formulae appear in one dimension when we find accumulation functions  $a(x)$  such that the  $n$ th power of the integral of  $a$  is itself exactly integrable. This is relatively easy: we search for analytically integrable  $f(x)$  for which  $f(x)^n = \int_0^x a = Q(x)$  is a vaguely realistic horizontal flux function.

EXAMPLE 1. A simplest example [5, 7] in this context is to suppose  $a(x) = a_0 > 0$  constant and require  $h = 0$  at a fixed position  $x = L$  (for a ice sheet symmetrical around a ridge at  $x = 0$ ). It should be understood, however, that the resulting  $Q$  *cannot be extended continuously into the ice-free region* as

$$Q(x) = \begin{cases} a_0 x, & \text{in ice,} \\ 0, & \text{in ice-free region.} \end{cases}$$

Recall that in this case  $h(x)^{2+2/n} = h_0^{2+2/n} - 2((n+2)a_0/\Gamma)^{1/n} x^{1+1/n}$ , which follows from equation (2). Equivalently we may write

$$h(x) = C_2 (L^{1+1/n} - |x|^{1+1/n})^{n/(2n+2)},$$

where  $C_2 = (2^n(n+2)a_0/\Gamma)^{1/(2n+2)}$ , because  $h_0 = 2^{n/2n+2}((n+2)a_0/\Gamma)^{1/2n+2} L^{1/2}$  from the boundary condition  $h(L) = 0$ . The functions  $a, Q, h$  are plotted in figure 1 in the case  $n = 3$ .

EXAMPLE 2. By careful choice of a *piecewise* constant accumulation/ablation function  $a(x)$  we can modify the above example so that  $Q(x)$  is continuous [2, 7]. In fact, if we again suppose an ice ridge at  $x = 0$ , we can choose  $R \in (0, L)$  and define functions  $a, Q$  corresponding to two regions of constant accumulation, as follows:

$$a(x) = \begin{cases} a_0, & 0 < x < R, \\ a_1, & R < x < L, \end{cases}, \quad \text{and} \quad Q(x) = \begin{cases} a_0 x, & 0 < x < R, \\ a_1(x - R) + a_0 R, & R < x < L, \end{cases}.$$

For  $Q(L) = 0$  we require  $0 = a_1(L - R) + a_0 R$ . If  $a_0 > 0$  then  $a_1 < 0$ , and furthermore if  $R \rightarrow L$  then  $a_0 \rightarrow -\infty$ . The limiting  $h$  as  $R \rightarrow L$  is given in the previous example. Let  $n = 3$  for simplicity. From equation (2),

$$h(x)^{8/3} = h_0^{8/3} - \frac{3}{4} C_1 \begin{cases} a_0^{1/3} x^{4/3}, & 0 < x < R, \\ a_0^{1/3} R^{4/3} + a_1^{-1} ((a_1(x - R) + a_0 R)^{4/3} - (a_0 R)^{4/3}), & R < x < L. \end{cases}$$

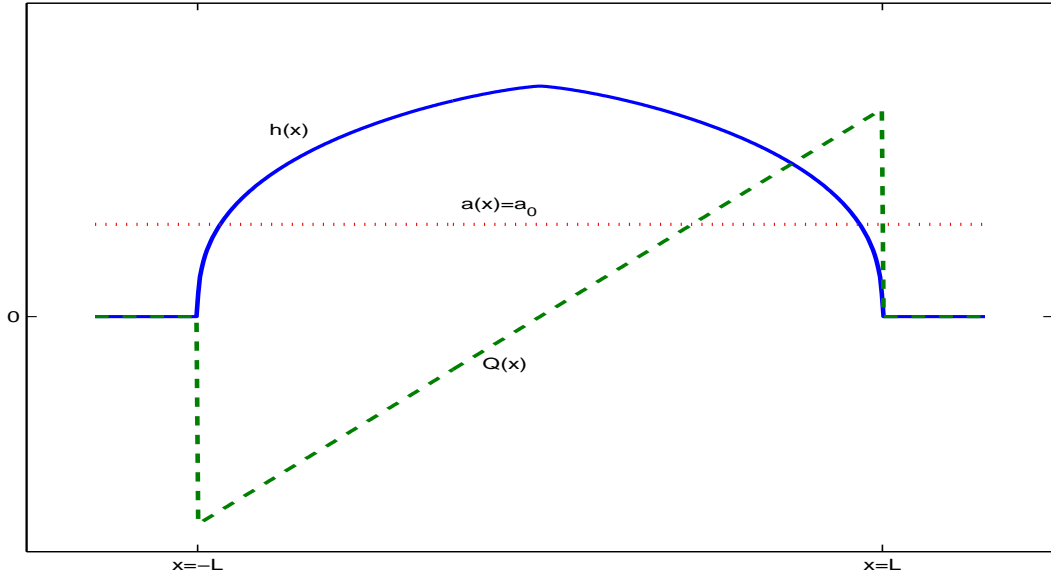


FIGURE 1. If the accumulation/ablation function  $a$  is constant and  $h = 0$  is imposed at  $x = L$  then the resulting flux  $Q$  cannot be extended continuously into the ice-free region. ( $n = 3$ .)

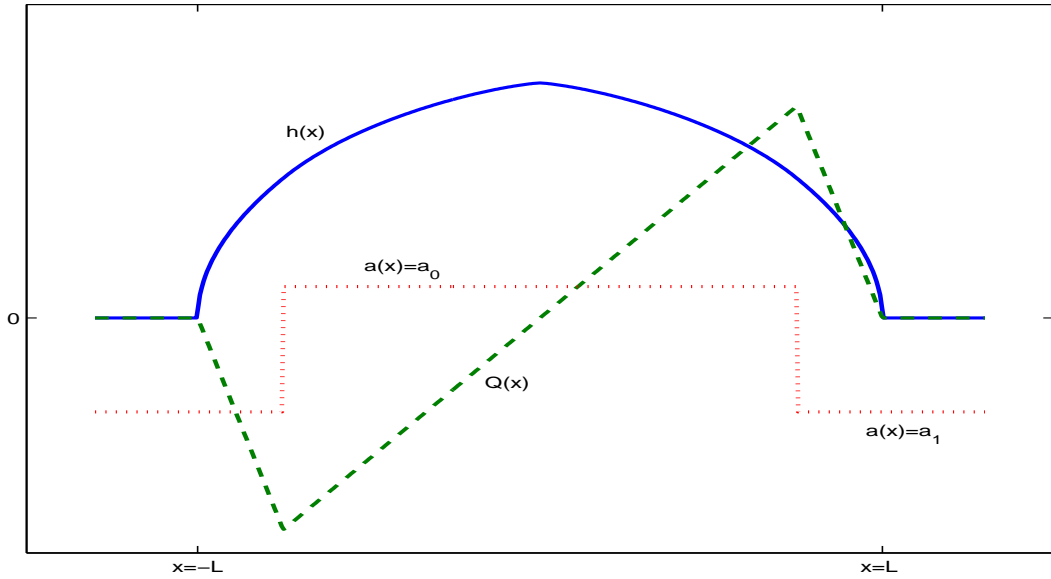


FIGURE 2. If the accumulation/ablation function  $a$  has the correct piecewise constant values then the flux  $Q$  can be extended continuously into the ice-free region. Of course,  $a$  is not continuous ( $n = 3$ ).

Note that because  $h(L) = 0$ , the constants are related as  $h_0^{8/3} = \frac{3}{4}C_1(a_0R)^{1/3}L$ . The functions  $a, Q, h$  are plotted in figure 2 for the case where  $R = \frac{3}{4}L$ . Unfortunately, in this example  $a$  is not continuous.

NOTE. The boundary behavior of these examples is worth recalling. In example 1, at the boundary  $x = L$ , there is accumulation. Thus, asymptotically,<sup>1</sup>  $h(x) \sim (L - x)^{n/(2n+2)}$ . For  $n = 3$  this is a  $3/8$  power. In example 2 there is ablation at the boundary and thus  $h(x) \sim (L - x)^{1/2}$  (independent of the value of  $n$  [7]). One might suppose the difference between  $1/2$  and  $3/8$  to be negligible but it is definitely not, in this context. Recall that in the shallow ice approximation [4], the effective shear stress at the base is  $\sigma = \rho g h |\nabla h|$ . For  $h(x) \sim (L - x)^q$  near the boundary  $x = L$  we see that  $\sigma \sim (L - x)^{2q-1}$ . In the ablation case, example 2,  $q = 1/2$  so  $\sigma$  is bounded. In example 1, by contrast, with  $n = 3$  and  $q = 2/(2n + 2) = 3/8$ ,  $\sigma \sim (L - x)^{-1/4}$  which is unbounded. According to the Glen law, the magnitude of the strain rates goes to infinity more strongly:  $\dot{\epsilon} \sim (L - x)^{-3/4}$ .

EXAMPLE 3. Again suppose  $n = 3$  for simplicity but choose

$$Q(x) = \alpha (\beta x^{1/3} + \gamma(L - x)^{1/3} - 1)^3$$

where  $\alpha, \beta, \gamma$  are constants to be determined. Note  $Q(x) \sim x$  for  $x$  near 0 and  $Q(x) \sim L - x$  for  $x$  near  $L$ . Determining  $\beta, \gamma$  by the conditions  $Q(0) = 0, Q(L) = 0$ , corresponding to an ice ridge at  $x = 0$  and the edge of the sheet at  $x = L$ , as in the previous two examples, we find  $\beta = \gamma = L^{-1/3}$ . Then  $a(x) = Q'(x) = \frac{\alpha}{L} \left( \left( \frac{x}{L} \right)^{1/3} + \left( 1 - \frac{x}{L} \right)^{1/3} - 1 \right)^2 \left[ \left( \frac{x}{L} \right)^{-2/3} - \left( 1 - \frac{x}{L} \right)^{-2/3} \right]$  is the corresponding accumulation function. In figure 3 we see  $a(x)$  decays roughly linearly for  $x > 0$ . In any case, from (2),

$$(3) \quad h(x) = h_0 \left[ 1 + 2\frac{x}{L} - \frac{3}{2}\left(\frac{x}{L}\right)^{4/3} + \frac{3}{2}\left(\left(1 - \frac{x}{L}\right)^{4/3} - 1\right) \right]^{3/8},$$

for  $x > 0$ , where  $\alpha, h_0$  are related by  $\alpha^{1/3} C_1 L = 2h_0^{8/3}$ ,  $h_0 = h(0)$ , and  $C_1 = (8/3)(5/\Gamma)^{1/3}$  as in (2). The functions  $a, Q, h$  are extended to  $[-L, 0]$  to make these even, odd, and even, respectively. The profiles  $a, Q, h$  are shown in figure (3) wherein  $Q$  is extended as an odd function and thus  $a, h$  are even. The boundary behavior of  $h$  is as in example 2.

EXERCISE. Example 3 can be extended to allow more control of the accumulation profile. In particular, in example 3 the accumulation has a strong peak at  $x = 0$  and this is avoidable. Show that if

$$Q(x) = \alpha (\beta \text{sign}(x)|x|^{j/3} + \gamma(L - x)^{1/3} - 1)^3$$

for  $0 < j < 3$  then the surface  $h(x)$  can be found by the method described above, and that the corresponding accumulation  $a$  satisfies  $a(x) \sim x^{j-1}$  as  $x \rightarrow 0$ . For example, the solution with  $j = 2$  has zero accumulation at  $x = 0$ .

### Radially-symmetric

We are ultimately interested in three spatial dimension ice sheets so we consider a circular ice sheet and construct the exact solution to the steady isothermal problem analogous to that in example 3. First, recall that if  $f = f(r)$  then  $\nabla f = f' \hat{\mathbf{r}}$  and  $\nabla \cdot (f \hat{\mathbf{r}}) = \frac{1}{r}(r f)'$ , where  $\hat{\mathbf{r}}$  is the standard unit radial vector field on the plane  $\mathbf{R}^2$ .

<sup>1</sup>By definition,  $f(x) \sim g(x)$  as  $x \rightarrow a$  if  $f(x)/g(x) \rightarrow C > 0$  as  $x \rightarrow a$ .

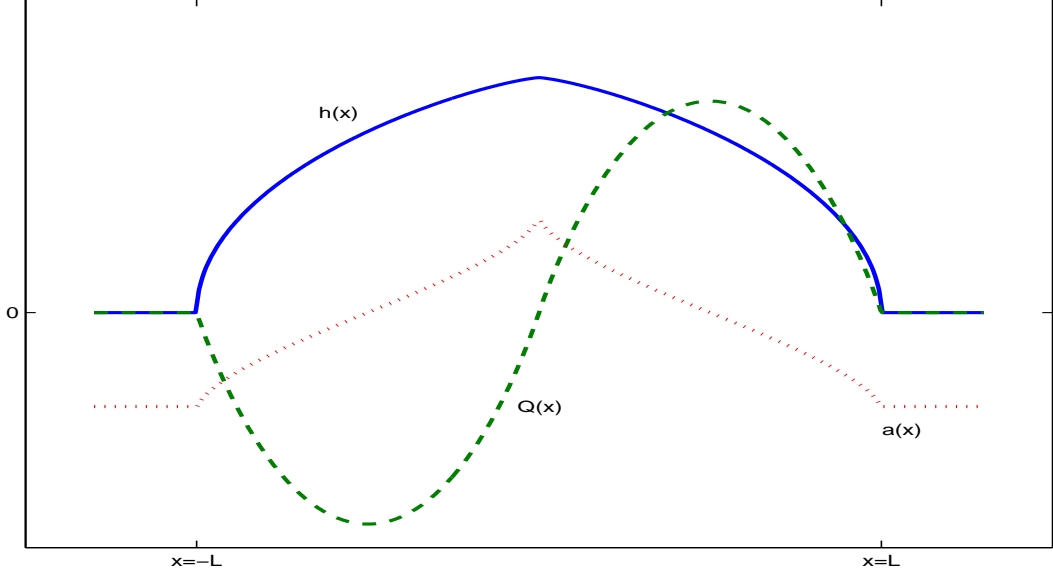


FIGURE 3. An exact, analytical ice sheet solution with smooth functions  $a$  and  $Q$  ( $n = 3$ ).

Suppose  $a, Q, h$  are radial functions. The steady, radially-symmetric shallow ice equation is  $a = \frac{1}{r}(rQ)'$  along with  $\frac{\Gamma}{n+2}h^{n+2}(-h')^n = Q$  if  $h' \leq 0$  for  $r > 0$ . It follows that

$$(4) \quad h(r)^{2+2/n} = h_0^{2+2/n} - C_1 \int_0^r \left( \frac{1}{\rho} \int_0^\rho \rho' a(\rho') d\rho' \right)^{1/n} d\rho$$

where  $C_1 = (2 + 2/n) \left(\frac{n+2}{\Gamma}\right)^{1/n}$  and  $h_0 = h(0)$  as before.

Thus we find an analytical solution whenever the  $(1/n)$ th power of  $Q(r)$  is integrable.

EXAMPLE 4. Suppose  $n > 1$  and seek  $Q$  of the form

$$Q(r) = \alpha (\beta r^{1/n} + \gamma(L-r)^{1/n} - 1)^n, \quad r > 0,$$

so that  $Q \sim r$  for  $r$  near zero and  $Q \sim L - r$  for  $r$  near  $L$ . If  $r = 0$  is an ice maximum and if  $r = L$  is the edge of the sheet, that is, if  $Q(0) = 0$  and  $Q(L) = 0$ , then  $\gamma = L^{1/n}$  and  $\beta = L^{1/n}$  respectively. Thus  $Q(r) = \alpha \left( \left(\frac{r}{L}\right)^{1/n} + \left(1 - \frac{r}{L}\right)^{1/n} - 1 \right)^n$ . From (4) above,

$$\begin{aligned} h(r)^{2+2/n} &= h_0^{2+2/n} - C_1 \int_0^r \alpha^{1/n} \left( \left(\frac{\rho}{L}\right)^{1/n} + \left(1 - \frac{\rho}{L}\right)^{1/n} - 1 \right) d\rho \\ &= h_0^{2+2/n} - \frac{C_1 \alpha^{1/n} L}{1 + 1/n} \left[ \left(\frac{r}{L}\right)^{1+1/n} - \left(1 - \frac{r}{L}\right)^{1+1/n} + 1 - (1 + 1/n) \frac{r}{L} \right]. \end{aligned}$$

If  $h(L) = 0$  then  $(1 + 1/n)h_0^{2+2/n} = C_1 \alpha^{1/n} L(1 - 1/n)$  so

$$(5) \quad h(r) = h_0 \left[ 1 - \frac{n}{n-1} \left( \left(\frac{r}{L}\right)^{1+1/n} - \left(1 - \frac{r}{L}\right)^{1+1/n} + 1 - (1 + 1/n) \frac{r}{L} \right) \right]^{n/(2n+2)}.$$

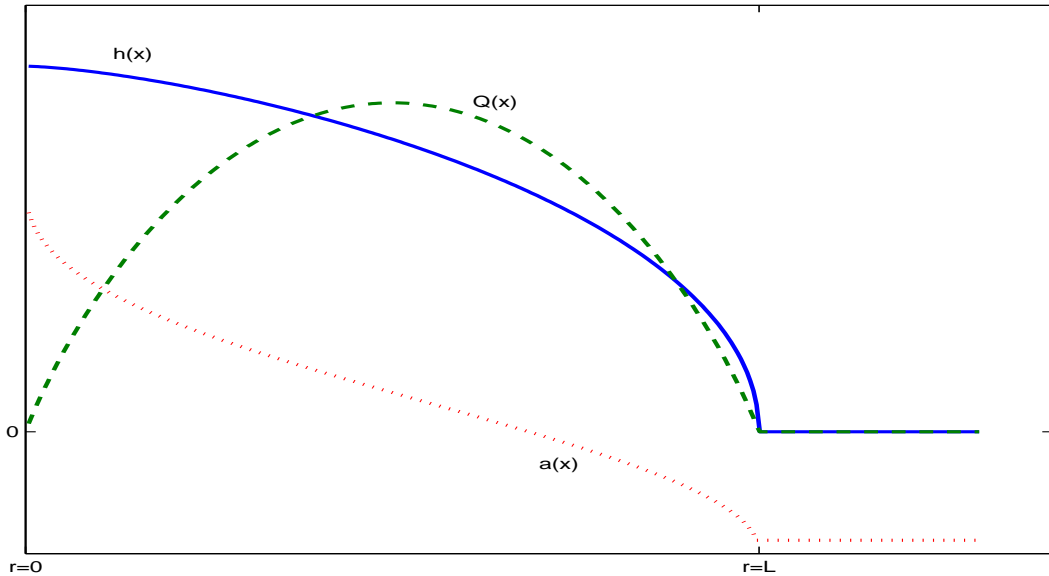


FIGURE 4. An exact radially-symmetric ice sheet solution for  $n = 1.8$  and  $Q(r) \sim ((r/L)^{1/n} + (1 - r/L)^{1/n} - 1)^n$ . Note  $a$  and  $Q$  are smooth and  $a$  decreases monotonically from the center of the sheet.

The accumulation function  $a$  corresponding to  $Q, h$  above is

$$a(r) = \frac{1}{r} (rQ)' = \frac{\alpha}{r} \left( \left( \frac{r}{L} \right)^{1/n} + \left( 1 - \frac{r}{L} \right)^{1/n} - 1 \right)^n + \frac{\alpha}{L} \left( \left( \frac{r}{L} \right)^{1/n} + \left( 1 - \frac{r}{L} \right)^{1/n} - 1 \right)^{n-1} \left( \left( \frac{r}{L} \right)^{\frac{1}{n}-1} - \left( 1 - \frac{r}{L} \right)^{\frac{1}{n}-1} \right).$$

Figures 4, 5 show profiles of  $a, Q, h$  for  $r > 0$  and  $n = 1.8, n = 4$  respectively.

In particular,  $h$  has two continuous derivatives in the interior of the sheet and  $Q, a$  are continuous everywhere. This is significant to the accuracy of numerical methods [6].

Example 1 (and its radial version) might be used to test the ability of a numerical model to approximate the shape of an advancing ice front in an accumulation zone, though of course in that case the ice would be building in advance of the front if the accumulation/ablation function were continuous. Example 2 has been used in numerical testing [2].

The above examples are used in numerical tests as follows. Regard the input to a steady-state simulation program as the act of specifying  $a = a(x, y)$ . If  $a$  is one of the examples given above then the program *should*, as output, produce an approximation to the corresponding function  $h$  above. This can be tested by coding both the exact  $a$ , as input, and the exact  $h$  for comparison to the computed output.

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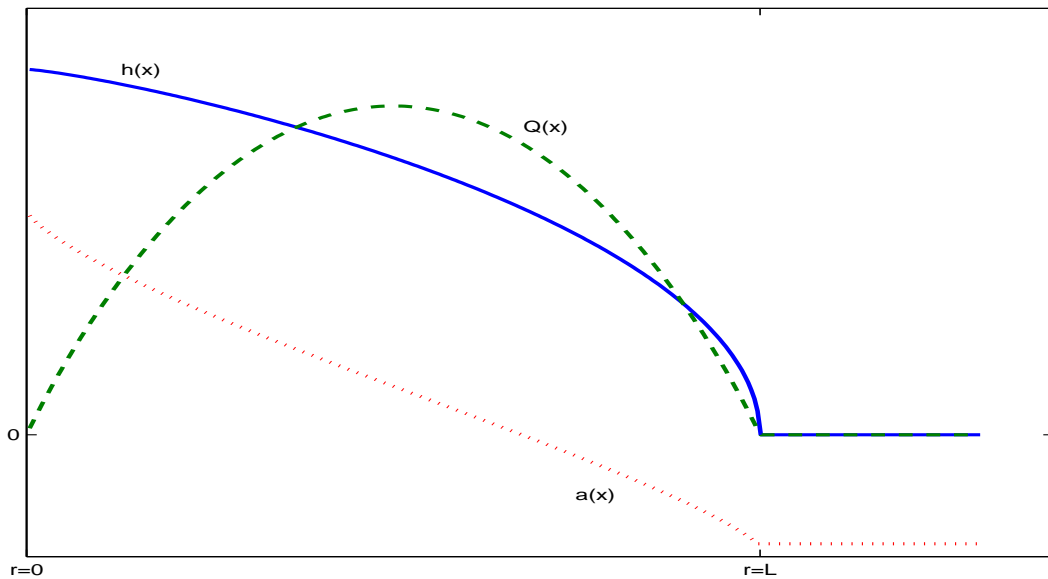


FIGURE 5. Compare to figure 4. Here  $n = 4$ . Note nearly linear decay for the accumulation function  $a$ .

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