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STABILITY OF UP- AND DOWN-MILLING USING CHEBYSHEV COLLOCATION METHOD

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ABSTRACT

The dynamic stability of the milling process is investigated through a single degree-of-freedom model by determining the regions where chatter (unstable) vibrations occur in the two-parameter space of spindle speed and depth of cut. Dynamic systems like milling are modeled by delay-differential equations (DDEs) with time-periodic coefficients. A new approximation technique for studying the stability properties of such systems is presented. The approach is based on the properties of Chebyshev polynomials and a collocation representation of the solution at their extremum points, the Chebyshev collocation points. The stability properties are determined by the eigenvalues of the approximate *monodromy matrix* which maps function values at the collocation points from one interval to the next. We check the results for convergence by varying the number of Chebyshev collocation points and by simulation of the transient response via the DDE23 MATLAB routine. The milling model used here was derived by Insperger *et al.* [14]. Here, the specific cutting force profiles, stability charts, and chatter frequency diagrams are produced for up-milling and down-milling cases for one and four cutting teeth and 25 to 100 % immersion levels. The unstable regions due to both secondary Hopf and flip (period-doubling) bifurcations are found which agree with the previous results found by other techniques. An in-depth investigation in the vicinity of the critical immersion ratio for down-milling (where the average cutting force changes sign) and its implication for stability is presented.

INTRODUCTION

One of the most important manufacturing processes is the milling process. The single degree-of-freedom model of the milling process leads to a delay differential equation (DDE) with time-periodic coefficients due to the time-varying nature of the forces on the cutting tool teeth. Although several analytical methods to find the stability boundaries for DDEs with constant coefficients exist, the stability criteria of the milling system cannot be given in a closed form. An approximation method is needed, which approximates the infinite dimensional monodromy operator with a finite dimensional matrix. Therefore, the stability map of the milling process as a function of the cutting parameters can be approximately determined.

Minis and Yanushevsky [1] used Fourier series expansions for periodic terms and determined the Fourier coefficients of related parametric transfer functions. Altintas and Budak [2] used a similar method except that they retained only the constant term in each Fourier series expansion of a periodic term. Davies et al. [3] and Zhao and Balachandran [4] examined how the periodic motions lost stability during partial immersion milling operations. Davies et al. [5] presented experimental results for milling operations with long, slender endmills, which indicate that the consideration of regenerative effects alone may not be sufficient to explain loss of stability of periodic motions for certain partial immersion operations. Davies et al. [6] analytically showed the existence of period-doubling instability lobes along with the traditional Hopf instability lobes in machining. The results were confirmed independently by Corpus and Endres [7], and by Insperger and Stepan [8,9]. These methods are not restricted to infinitesimal times in the cut. Bayly et al. [10,11] extended the previous approaches by the use of time finite element analysis. This approach also led to stability analysis of a discrete map, but the requirement of small time in the cut was relaxed. Analytical and experimental results were obtained for a 1-DOF system. Most of the stability results obtained by using the above mentioned approximation methods and the methods used by the researchers agree with each other.

Insperger et al. [12] also performed a frequency analysis to obtain the stability conditions of time-periodic DDEs from which they discovered that chatter frequencies (secondary Hopf bifurcation and period doubling bifurcation) occur at the stability boundaries. They also analyzed the stability conditions of up- and down-milling operations [13,14] using the semi-discretization method [15] and the temporal finite element method. The study was restricted to a 1-DOF milling model that has the cutting tool carrying a single flute. Bayly et al. [16] extended the previous work to a two degree-of-freedom model.

The present work represents the implementation of a new approximation technique based on Chebyshev collocation. It solves the time periodic linear DDEs with multiple integer delays and piecewise smooth coefficients [17]. This method evolved from the methods developed by Sinha and Wu [18] to solve periodic ODEs using the Chebyshev polynomial approximation and by Butcher et al. [19] to obtain the monodromy matrix for time-periodic DDEs with smooth coefficients by Chebyshev polynomial expansion of the solution. The collocation method is shown in [17] to be spectrally accurate to initial value problems. It gives an approximation to the compact monodromy operator of the DDE, whose eigenvalues converge spectrally to the exact Floquet multipliers. The method generalizes and extends to the periodic coefficients case the linear multi-step methods and pseudospectral techniques introduced in [20,21], and leads to exponentially fast convergence about the Floquet multipliers. It is flexible for systems with multiple degrees of freedom and it produces stability charts with high speed and accuracy in a given parameter range. In this work, stability charts and frequency diagrams are produced for up-milling and down-milling cases of several cutting teeth and 25 to 100 % immersion levels using the Chebyshev collocation method. The unstable regions due to both secondary Hopf and flip bifurcations are found which agree with the results found by other techniques in the literature. An investigation in the vicinity of the critical immersion ratio for down-milling (where the average cutting force changes from negative to positive) and its implication for stability is presented.

MECHANICAL MODEL OF MILLING

We use the same single degree-of-freedom milling model as in [14], to which the reader is referred for additional details in the derivation. The tool is assumed to be flexible in the feed direction only. A summation of forces acting on the tool in that direction produces the equation of motion

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \frac{F(t)}{m}, \quad (1)$$

where m is the modal mass, ζ is the damping ratio, ω_n is the natural angular frequency, and $F(t)$ is the total cutting force in the feed direction on all engaged cutting teeth. The force on the p^{th} tooth is given by

$$F_p(t) = g_p(t)(-F_{tp}(t)\cos\theta_p(t) - F_{np}(t)\sin\theta_p(t)) \quad (2)$$

where $g_p(t)$ acts as a switching function. It is equal to one if the p^{th} tooth is active and zero if it is not cutting. $\theta_p(t)$ is the cutter angle of the pth tooth as it rotates. The cutting force components are the product of the tangential and normal linearized cutting coefficients K_t and K_n, respectively, the nominal depth of cut b, and the chip width $w_p(t)$ as

$$F_{tp}(t) = K_t b w_p(t), \; F_{np}(t) = K_n b w_p(t) \quad (3)$$

where

$$w_p(t) = f\sin\theta_p(t) + [x(t) - x(t-\tau)]\sin\theta_p(t) \quad (4)$$

depends on the feed per tooth f, the current and delayed position of the tool, and $\theta_p(t)$. Here, $\tau = 60/N\Omega$ [s] is the tooth pass period, Ω is the spindle speed given in rpm, and N is the number of teeth.

A summation over the total number N of cutting teeth, and the substitution of equations (3-4) into equation (2) yields

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = -\frac{bh(t)}{m}[x(t) - x(t-\tau)] - \frac{bf_0(t)}{m} \quad (5)$$

where

$$h(t) = \sum_{p=1}^{N} g_p(t)[K_t\cos\theta_p(t) + K_n\sin\theta_p(t)]\sin\theta_p(t) \quad (6)$$

is the τ –periodic specific cutting force variation,

$$f_0(t) = \sum_{p=1}^{N} g_p(t)[K_t\cos\theta_p(t) + K_n\sin\theta_p(t)]\, f\sin\theta_p(t) \quad (7)$$

and the angular position of the tool is $\theta_p(t) = (2\pi\Omega/60)t + p2\pi/N$, where Ω is given in rpm.

A solution to equation (5) is assumed of the form

$$x(t) = x_p(t) + \xi(t) \tag{8}$$

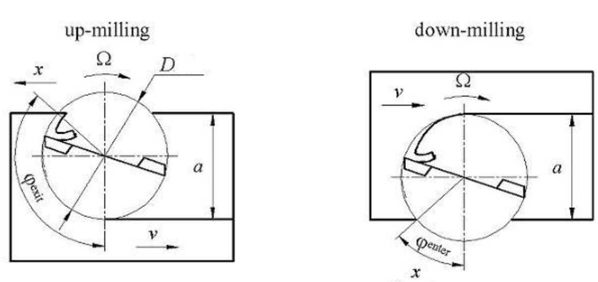
where $x_p(t) = x_p(t+\tau)$ is the unperturbed τ-periodic motion, and $\xi(t)$ is the perturbation which vanishes when no regenerative chatter vibrations are present. Substitution of equation (8) into equation (5) yields

$$\ddot{\xi}(t) + 2\zeta\omega_n\dot{\xi}(t) + \omega_n^2\xi(t) = -\frac{bh(t)}{m}[\xi(t) - \xi(t-\tau)] \tag{9}$$

This is the linear variational DDE model used in this paper (and in [14]). Stability of the $\xi(t)=0$ solution in equation (9) implies the stability of the ideal (chatter-free) motion $x_p(t)$.

UP-MILLING AND DOWN-MILLING

The relationship between the direction of tool rotation and the feed defines two types of partial immersion milling operations: the up-milling and down-milling operations. Both operations work in a similar way except that the rotation of the cutting tool is in the opposite direction. However the dynamics and stability properties are different. Partial immersion milling operations are characterized by the number *N* of teeth and the radial immersion ratio '*a*/*D*', where *a* is the radial depth of cut, and *D* the diameter of the tool. We can differentiate up-milling from down-milling by knowing the angles of contact made by a particular tooth inside the workpiece. The specific cutting force variation *h(t)* in equation (6) depends on the screen function for the *p*th tooth which is defined as $g_p(t) = 1$ if $\theta^{enter} < \theta_p(t) < \theta^{exit}$ and $g_p(t) = 0$ otherwise. The entry and exit angles can be found from the figure below [14] as $\theta^{enter} = 0$ and $\theta^{exit} = \cos^{-1}(1-2a/D)$ for up-milling, while for down-milling the angles are $\theta^{enter} = \cos^{-1}(2a/D-1)$ and $\theta^{exit} = \pi$.



The specific cutting force for up- and down-milling for immersion ratios of 0.25, 0.5, 0.75, and 1.0 are shown in Figures 1-4 for the cases of one and four cutting teeth.

While the stability charts for up- and down-milling for a single cutting tooth were presented in [14], we have produced charts for 1,2,4, and 8 teeth. Here we show up- and down-milling results for 1 and 4 teeth for the above immersion ratios.

CHEBYSHEV COLLOCATION APPROXIMATION

The Chebyshev collocation approximation method used to solve the milling problem and obtain the stability diagrams is based on the properties of the Chebyshev polynomials. The standard formula to obtain the Chebyshev polynomial of degree *j*, which is denoted by $T_j(t)$ is

$$T_j(t) = \cos j\theta,\ \theta = \arccos(t),\ -1 \le t \le 1 \tag{10}$$

The *Chebyshev collocation points* are unevenly spaced in the given domain corresponding to extreme points of the Chebyshev polynomial. We can visualize these points as the projections on the domain [-1,1] of equispaced points on the upper half of the unit circle as

$$t_j = \cos(j\pi/(m-1)), \qquad j = 0, 1, \ldots\ldots, m-1 \tag{11}$$

A spectral differentiation matrix for *m* Chebyshev collocation points is obtained by interpolating a polynomial through the function values at the collocation points, differentiating that polynomial, and then evaluating the resulting polynomial at the collocation points. As shown in [22], the differentiation matrix *D* has the following form:

$$D_{11} = \frac{2(m-1)^2+1}{6},\ D_{mm} = -\frac{2(m-1)^2+1}{6},$$

$$D_{jj} = \frac{-t_{j-1}}{2(1-t_{j-1}^2)}, \qquad j = 2, \ldots\ldots, m-1, \tag{12}$$

$$D_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(t_{i-1}-t_{j-1})}, \qquad i \neq j,\ i, j = 1, \ldots\ldots, m$$

$$c_i = \begin{cases} 2, & i = 1, m \\ 1, & otherwise \end{cases}$$

The dimension of *D* is $m \times m$ where *m* is the number of Chebyshev points. If I_n is the *n x n* identity, then we also define a dimension *mn x mn* differentiation matrix using the Kronecker product operation as

$$\hat{D} = D \otimes I_n, \tag{13}$$

Now consider a linear, time periodic system of *n* DDEs with fixed delay $\tau > 0$,

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau)$$
$$x(t) = \phi(t)\,, \qquad t_0 - \tau \le t \le t_0 \qquad (14)$$

where $x(t)$ is a $n\times 1$ state vector, $A(t) = A(t+T)$ and $B(t) = B(t+T)$ are $n\times n$ periodic matrices, $\phi(t)$ is an $n\times 1$ initial vector function in the interval $[-\tau, 0]$. Assuming the delay is equal to the period ($\tau = T$, which is not a necessary assumption for the procedure), if $\Phi(t)$ is the fundamental solution matrix to the non-delay part of (14) and $\Psi(t)$ is the fundamental solution matrix for the adjoint system such that $\Phi^{-1}(t) = \Psi^T(t)$, the infinite-dimensional monodromy operator for a periodic DDE system can be defined as [23]

$$(Ux)(t) = \Phi(t)\{x(\tau) + \int_0^t \Psi(s)^T B(s)x(s)ds\} \qquad (15)$$

which maps continuous functions from the interval [0,*T*] back to the same interval, i.e., $U: \; C[0,T] \to C[0,T]$. If the maximum of the modulus of the eigenvalues (Floquet multipliers) of the monodromy operator *U* is less than 1, then the system is said to be stable. It is impossible to numerically find all the eigenvalues of the infinite dimensional *U* matrix. However, we use the Chebyshev collocation approximation method to reduce the size of the *U* matrix to a finite dimension, whose spectral radius decides the stability. Because of the compactness of the *U* matrix, all of the neglected eigenvalues are guaranteed to be clustered about the origin and thus do not influence the stability.

Solving the numerical approximation of equation (14) using the Chebyshev collocation method will give an approximation to the monodromy operator in equation (15) [17]. Finding the approximate solution by *knowing the function values at different points in a given interval* is the basic idea of collocation. First, let $\{\phi_j\}$ and $\{v_j\}$, *j=1,…,m* be sets of function values at *shifted* Chebyshev collocation points in the interval [0,T] where the points are ordered *right to left* as in equation (11) and [22]. The $\{\phi_j\}$ are given values of the initial function $\phi(t)$ in the normalized interval $t \in$[-T,0] and the $\{v_j\}$ are values of the solution $x(t)$ to be found in the normalized interval $t \in$[0,T]. Note that the matching condition at *t*=0 requires that $\phi_1 = v_m$. Then by the method of steps we can obtain the $\{v_j\}$ as $\{v_j\} = U\{\phi_j\}$ for a finite matrix *U* which approximates the monodromy operator.

To obtain *U*, we write equation (14) in the collocation approximation form as

$$D^{\wedge}\{v_j\} = M_A^{\wedge}\{v_j\} + M_B^{\wedge}\{\phi_j\} \qquad (16)$$

The matrix $D^{\wedge}$ is obtained from $\tilde{D}$ by modifying the last *n* rows as [0_n 0_n 0_n … I_n] where 0_n and I_n are *n x n* null and identity matrices and then scaling to account for the shift $[-1,1] \to [0,T]$ by multiplying the resulting matrix by 2/*T*. The patterns of the $M_A^{\wedge}, M_B^{\wedge}$ matrices are

$$M_A^{\wedge} = \begin{bmatrix} A(t_1) & & & & & \\ & A(t_2) & & & & \\ & & A(t_3) & & & \\ & & & . & & \\ & & & & A(t_{m-1}) & \\ 0_n & 0_n & . & . & 0_n & 0_n \end{bmatrix}, \qquad (17)$$

and

$$M_B^{\wedge} = \begin{bmatrix} B(t_1) & & & & & \\ & B(t_2) & & & & \\ & & B(t_3) & & & \\ & & & . & & \\ & & & & B(t_{m-1}) & \\ I_n & 0_n & . & . & 0_n & 0_n \end{bmatrix} \qquad (18)$$

where $A(t_i)$ means *A(t)* evaluated at the i^{th} shifted collocation point. Here the hat ^ next to the operators refers to the modification of the last *n* rows to account for the *n* matching conditions between successive intervals (including the modification to $D^{\wedge}$ above). Therefore, we get the approximation to the monodromy operator as

$$U = \left[D^{\wedge} - M_A^{\wedge} \right]^{-1} M_B^{\wedge} \qquad (19)$$

If *m* is the number of collocation points and *n* is the size of the DDE system, then the size of the *U* matrix will be *mn* × *mn*. We can achieve higher accuracy by increasing the value of *m*. In [17] it is shown in an *a posteriori* sense that if *A(t)* and *B(t)* are sufficiently smooth then the approximate eigenvalues (Floquet multipliers) of *U* converge to the exact eigenvalues of *U* in equation (15) at an exponential rate.

STABILITY CHARTS AND FREQUENCY DIAGRAMS

Stability charts are determined by using the Chebyshev collocation method to analyze the monodromy operator as it depends on parameters. We will consider a series of milling processes like up-milling and down-milling, varying immersion ratios, and varying number of cutting teeth. Since the specific cutting force variation *h*(*t*) is independent of the spindle speed of the tool, we assume the spindle speed Ω is

3300 rpm to plot $h(t)$. Experimentally identified parameters given in [15] are used to construct the stability charts: m = 2.573 kg, $\zeta = 0.0032$, $\omega_n = 920.02$ Hz, $K_n = 2.0\times10^8$ N/m ^{2} and $K_t = 5.5\times10^8$ N/m ^{2}. Stability charts are constructed with parameters being the spindle speed Ω (ranging from 2000 to 25000 rpm) and the chip thickness b (ranging from 0 to 5 mm). MATLAB software is used for producing the stability chart using the collocation method. The dimension of the monodromy operator for all cases is 80×80. However it is possible to produce the same stability chart by taking a lower dimensional (e.g. 40×40) monodromy matrix for most of the cases (except for the critical immersion ratio cases). We chose 300×300 grid points in the parameter plane. The computational specifications used to run the MATLAB programs are: Intel Pentium IV, processor speed 1.5 GHz, RAM 1.02 GB. It takes approximately 90-120 minutes to obtain each stability diagram.

The stability analysis is based on the determination of the relevant characteristic multiplier using the collocation method. We use bifurcation theory to explain the type of instability. For the μ =1 (fold bifurcation) case, it can be shown that this bifurcation cannot occur in the milling equation. For secondary Hopf bifurcation, $|\mu|$=1 and $\lambda = i\omega$ is purely imaginary where ω =Im(ln μ)/ τ . In this case, the chatter frequencies are determined from ω, which is also the positive angle made by the characteristic multiplier in the complex plane. Since the complex exponential function is periodic, the logarithmic function is not unique in the plane of complex numbers. This raises the possibility of multiple chatter frequencies. These chatter frequencies can be observed while doing experiments on the milling machines. These frequencies can be measurable and comparable with the theoretical results.

Chatter frequency diagrams are constructed in Figures (1-4) by considering the characteristic multipliers obtained at the stability boundary and using the formulation in [12]. However, if the characteristic multipliers are found using the collocation method, then the equations for Hopf frequencies in [12] must be altered by dividing by the factor τ due to the normalization used in the collocation method. Therefore, the Hopf frequencies are given as

$$f_H = \left(\pm\frac{\omega}{2\pi} + n\right)\frac{N\Omega}{60} \text{ [Hz]}, \quad n = \dots,-1, 0, 1,\dots \tag{20}$$

where τ is given in sec. and Ω in rpm. For the period doubling case (μ = -1), the characteristic exponent is $\lambda = (\ln(-1))/\tau$ or we substitute angle $\omega = \pi$ into (20) as

$$f_{PD} = \left(\frac{1}{2} + n\right)\frac{N\Omega}{60} \text{ [Hz]}, \quad n = \dots,-1, 0, 1,\dots \tag{21}$$

We check the response at some of the parameter points in the stability charts using the MATLAB routine DDE23 [24]. If the solution of the given system decays as time goes to infinity, then the system is said to be stable at the given parameter points; otherwise the system is unstable. Using this concept we pick three parameter points from the stability charts shown in Figures 1 and 4 and, using DDE23, we check whether those parameter points are stable or unstable. The results shown in Figures 5-6 agree with the stability charts obtained by the collocation method (see the locations of characteristic multipliers obtained by using the collocation method).

DISCUSSION

For some cases in the milling process, we can notice a drastic change in the stability charts just by changing the immersion ratio. Consider the stability charts of the down milling single tooth case shown in Figure 2, where the order of Hopf bifurcation stability lobe (‘U’ shaped) and flip bifurcation stability lobe (‘V’ shaped) is switched by changing the immersion ratio. Stability charts drawn for different immersion ratios between 0.62 to 0.71are shown in Figure 7, to illustrate what really happens to the stability diagrams between these immersion ratios. We can see that the milling case with immersion ratios 0.63 to 0.68 has larger stability region compared to any other milling case for the given spindle speed Ω and is fully stable for the spindle speed range of 9000 to 16000 r.p.m. We can also notice that the immersion ratios above 0.663 have a positive average specific cutting force variation $h(t)$ value, whereas for lower immersion ratios, the value is negative. This is one of the reasons explaining the drastic change in the stability conditions near the critical immersion ratio. For the negative depth of cut case, with immersion ratios less than the critical immersion ratio, the corresponding stability diagrams reveal information about obtaining the stability region for positive chip thickness by knowing the unstable region for negative chip thickness. Note that the above theory is applicable to only Hopf type stability lobes (i.e., the flip type lobes do not change drastically).

In Figures 1-4, the similarities and differences between upmilling and down milling can be clearly observed. The flip (period doubling) lobes, for example, vary in size but are located more or less at the same spindle speed range (around 16000 to 22000 r.p.m). This is not true for Hopf lobes. For low immersion upmilling, the Hopf lobes are located to the left of flip lobes, while the downmilling cases show this special duality or mirror symmetry for immersion ratios with 0.5 or less. An explanation for these interesting results is as follows: The flip lobes are related to the impact effects of entering and leaving the workpiece material. While these are more or less independent of the sense (up or downmilling) of the milling, this is not the case for the Hopf lobes. These flip lobes occur for lower immersions and lower number of cutting teeth cases. High speed milling operations can be stabilized simply by changing to downmilling from upmilling at certain wide high speed parameter domains (9000 to 16000 rpm). This is where the critical immersion ratio range of 0.63 to 0.68 for down-milling is so important, because it has a higher stability region than any other case.

For the multiple cutting teeth upmilling case (Figure 3), the presence of idle time (i.e., if $h(t)$ is zero) leads to flip lobes in the stability chart. According to the assumption made earlier, that any p^{th} tooth follows the same cutting profile as the first cutting tooth, leads us to the conclusion that for all even numbers of teeth with full immersion (except $N = 2$), we will have constant specific cutting force variation that makes this milling case look similar to the turning operation. Also the stability charts for milling and turning cases look the same. The DDE23 results (Figures 5-6) and Chebyshev collocation results agree with each other. For the specific cutting force variation (Figures 1-4), the approximation of $h(t)$ using Chebyshev points gives reasonably good results with similar relative errors compared to the other methods which use equispaced points. The number of Chebyshev points should be large enough to get reasonably accurate stability charts. Thus, also the Chebyshev collocation method is exponentially convergent for *smooth* coefficients [17], the presence of discontinuities in the specific cutting force variation leads to a higher minimum number of points *for the milling problem* than what would normally be expected. (The suggested minimum number for m is 20, while near the critical immersion ratio in Figure 7 we use $m = 80$.)

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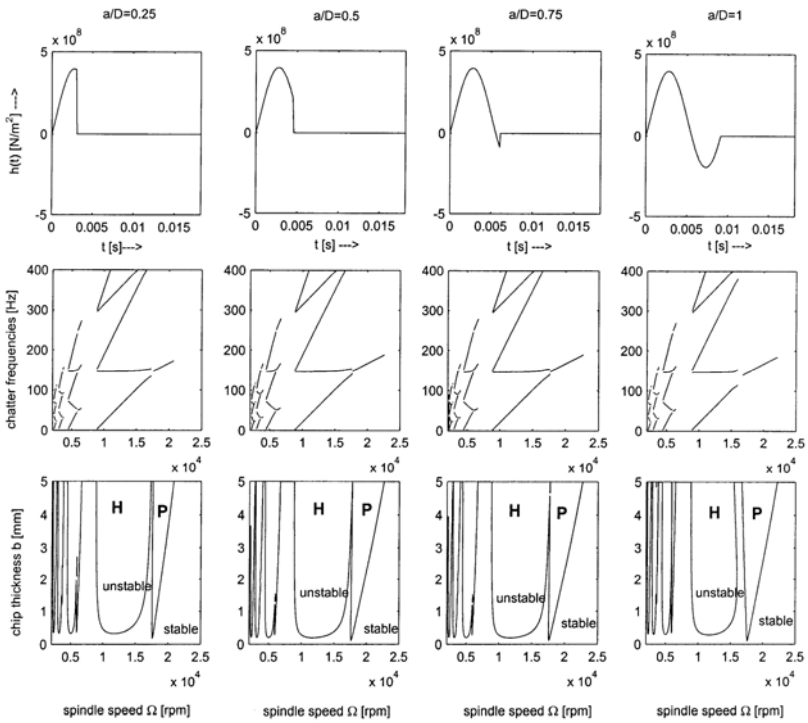

Figure 1. Up-milling, number of cutting teeth N = 1, specific cutting force variation diagrams, frequency diagrams and stability diagrams for varying immersion ratios a/D=0.25, 0.5, 0.75, 1

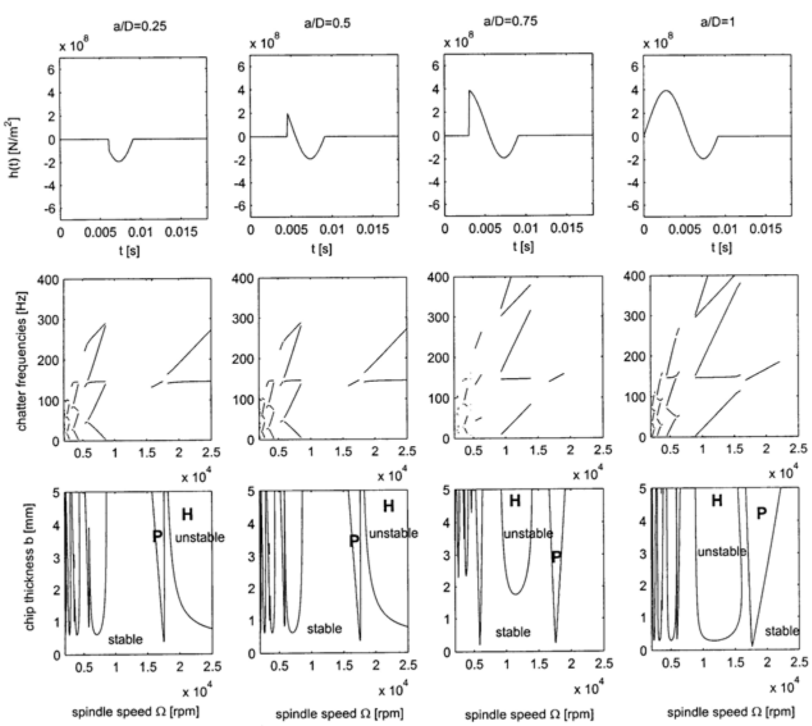

Figure 2. Down-milling, number of cutting teeth N = 1, specific cutting force variation diagrams, frequency diagrams and stability diagrams for varying immersion ratios a/D=0.25, 0.5, 0.75, 1

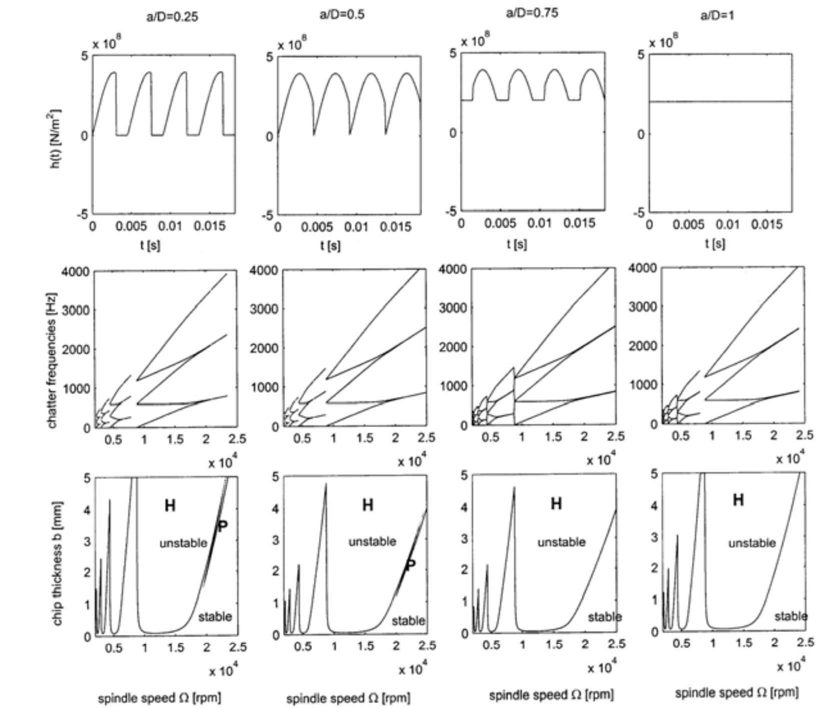

Figure 3. Up-milling, number of cutting teeth N = 4, specific cutting force variation diagrams, frequency diagrams and stability diagrams for varying immersion ratios a/D=0.25, 0.5, 0.75, 1

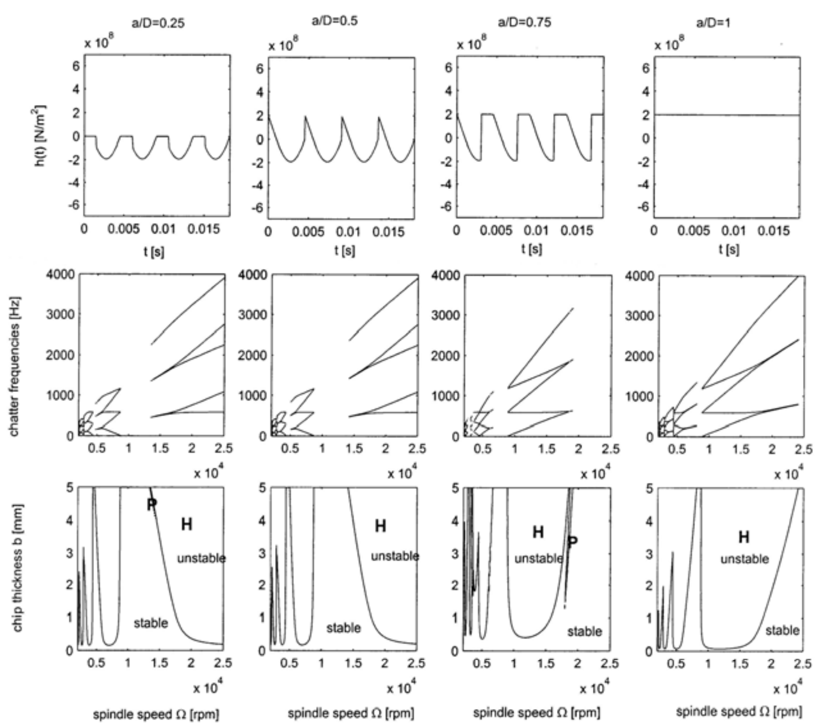

Figure 4. Down-milling, number of cutting teeth N = 4, specific cutting force variation diagrams, frequency diagrams and stability diagrams for varying immersion ratios a/D=0.25, 0.5, 0.75, 1

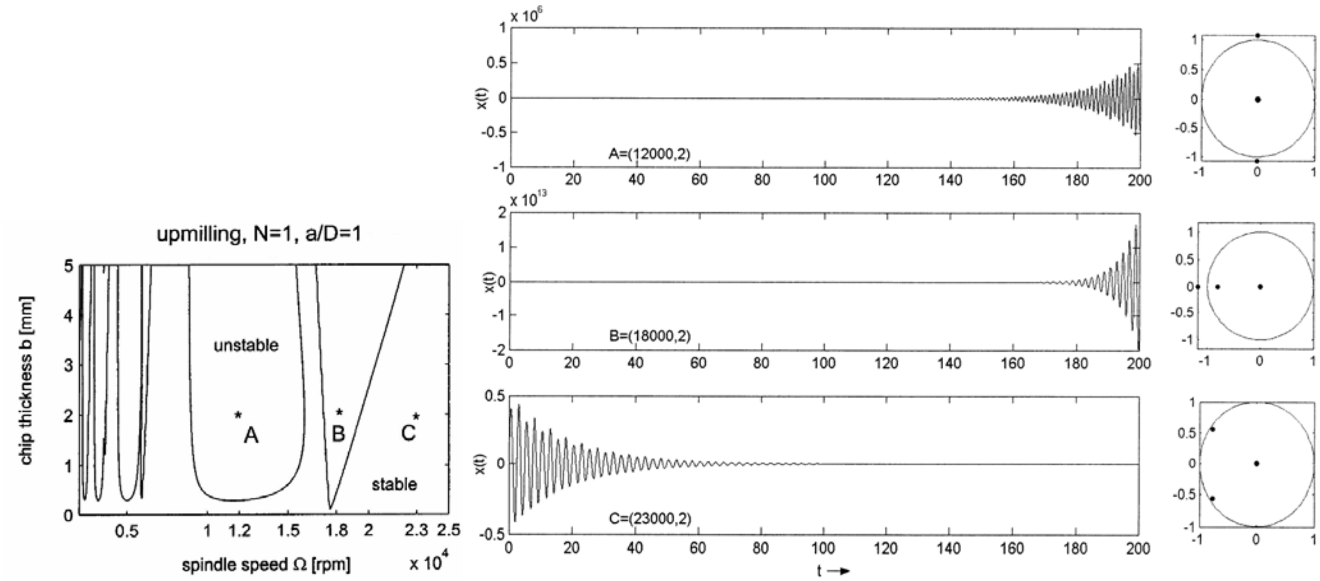

Figure 5. DDE23 results for the parameter points A, B and C picked from the stability diagram of up-milling with N = 1, a/D = 1 and collocation results for finding the locations of characteristic multipliers

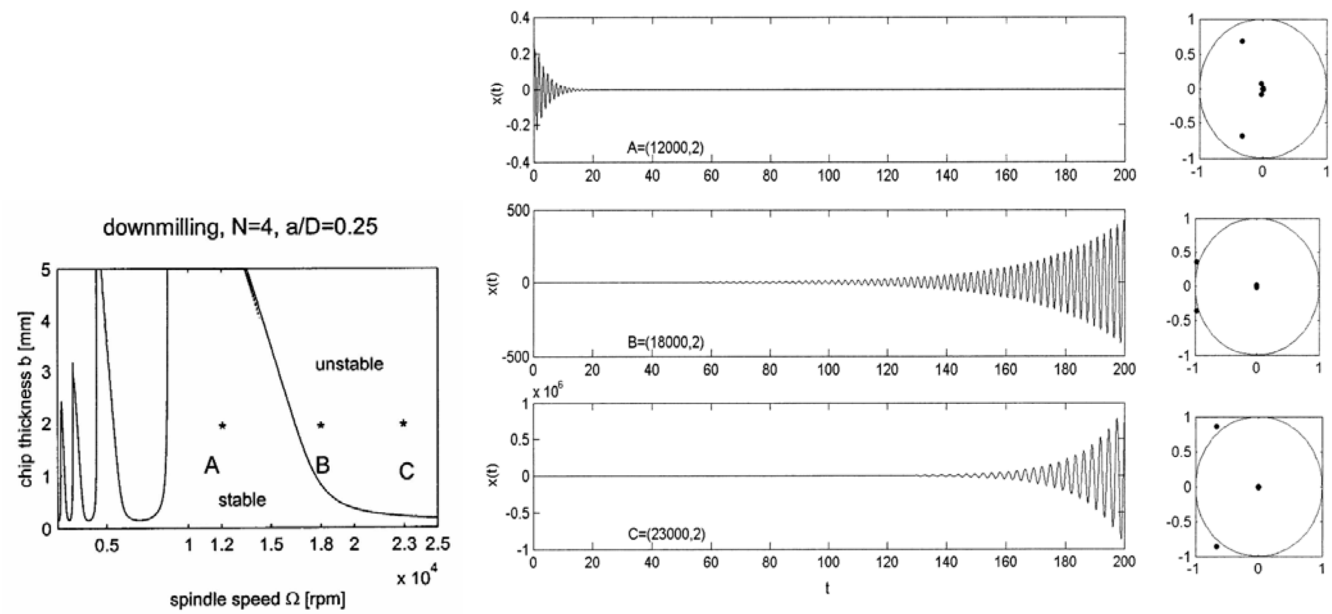

Figure 6. DDE23 results for the parameter points A, B and C picked from the stability diagram of down-milling with N = 4, a/D = 0.25 and collocation results for finding the locations of characteristic multipliers

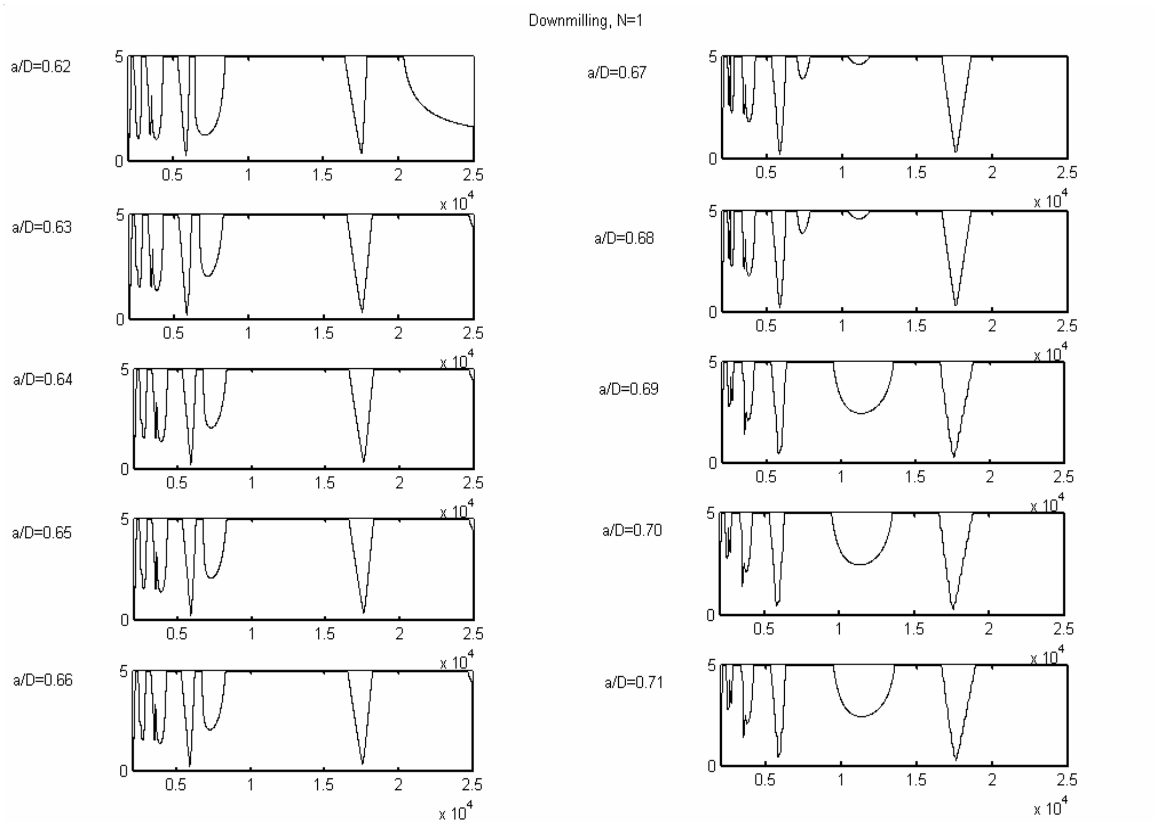

Figure 7. Critical immersion ratios for down-milling, N = 1, collocation points m = 80, parameter plane 300×300 grid points