DIRAC OPERATORS AS "ANNIHILATION OPERATORS" ON RIEMANNIAN MANIFOLDS

ED BUELER

Abstract. The following two situations are shown to be equivalent.

I. The commutation relation $[A, A^*] = \alpha$ holds on the span of excited states of the form $(A^*)^k \zeta$. Here A is a Dirac operator acting in a weighted Hilbert space of sections of a Dirac bundle S over a Riemannian manifold M, ζ is a vacuum state, and $\alpha > 0$ is a constant.

II. There exists a scalar solution h on all of M to the simultaneous equations $\triangle h = \alpha$ and $\alpha h + \frac{1}{2}|\nabla h|^2 = 0$.

The two situations are connected by writing $e^{2h}\, dx$ for the weight (measure) for the Hilbert space of sections of S.

1. Introduction

The quantum harmonic oscillator Hamiltonian is equal, up to an additive constant, to a number operator $N = a_i^* a_i$ for annihilation operators a_i and creation operators a_i^* which satisfy the commutation relations $[a_i, a_j^*] = \delta_{ij}\hbar$. It follows that N has, up to a multiplicative constant, spectrum equal to the nonnegative integers. The theory of these operators has a famous representation in which the a_i are the basic first–order derivatives $\frac{\partial}{\partial x^i}$ on the Euclidean space $\mathbf{R}^n$. One constructs this representation from the position representation via the natural unitary map between $L^2(\mathbf{R}^n, \text{Lebesgue measure})$ to $L^2(\mathbf{R}^n, \text{Gauss measure})$.

Recent discoveries in the harmonic analysis of a compact Lie group G, originating with [Gro93], suggest that on G the obvious first–order operators on functions can be regarded as annihilation operators in some sense. They act in a Hilbert space where heat kernel measure plays the same role as Gauss measure does on $\mathbf{R}^n$. Specifically, let $a_X = X$ for X in the Lie algebra of G. Suppose a_X acts in $L^2(G, \rho_t\, dx)$ where ρ_t is the heat kernel at the identity of G and $t > 0$ is fixed. Then $a_X^* = -X - X(\log \rho_t)$ from which $[a_X, a_Y^*] = -[X, Y] - XY(\log \rho_t)$ follows. Thus the annihilation and creation operators do not satisfy the original commutation relations, except in the $G = \mathbf{R}^n$ case. Nevertheless these operators are one aspect of a fruitful analysis on Lie groups. See [Hal01] and the references therein.

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In the above context, annihilation operators arise as the natural first derivative operators. A general (connected and complete) Riemannian manifold M^n obviously possesses no preferred finite–dimensional space of first–order operators acting on functions which might serve as annihilation operators. There exist, however, globally defined and natural first–order operators on *vector bundles* over M. In fact, we will see that we can answer the question:

> For a Dirac operator D acting on sections of a Dirac bundle S, what conditions on M, S, and a measure $\nu\, dx$ characterize the situation $[D, D^*] =$ (positive constant) on excited states generated from a vacuum state by applying powers of D^*?

The question assumes D acts in the Hilbert space $L^2(S, \nu\, dx)$ of sections of S, where ν is some smooth positive function, and that the adjoint D^* is calculated in that space. Also, dx is the Riemann–Lebesgue measure.

As an example of a Dirac bundle and operator, consider the bundle of differential forms with $D = d+\delta$. The spin bundles ([LM89]) are also Dirac bundles, of course.

The answer to the question is that if $h = \frac{1}{2}\log\nu$ satisfies equations (1) and (2) below then $[D, D^*] =$ (constant) on the span of excited states, defined below. Conversely, these conditions on h are necessary. These results (see theorems 6 and 7) are the main ones of the paper.

Assume h solves (1) and (2). We construct, from $N = \frac{1}{2}D^*D$ by the natural unitary transform, the Schrödinger operator $\frac{1}{2}D^2 - \alpha h - \frac{\alpha}{2} + \gamma$ acting on the unweighted L^2 sections of S. This operator is self–adjoint (see theorem 10). A spectral result, showing that it is a "partial" number operator under some geometrical conditions, appears in corollary 12.

The author's original motivation was not from the Lie groups context cited above, but rather from a desire to understand the spectral properties of a Hodge operator weighted by heat kernel measure ([Bue99],[GW00]). Note that the small–time limit of the heat kernel solves equation (2) in the following sense: Let $h(x) = \frac{1}{2}\lim_{t\to 0^+} t\log\rho_t(x_0, x)$ where ρ is the heat kernel on M and $x_0 \in M$. By a result of Varadhan [Var67], h solves (2) with $\alpha = \frac{1}{2}$ since in fact $h(x) = -\frac{1}{4}d(x_0, x)^2$ where d is the Riemannian distance. Note h solves equation (1) *with* α *replaced by* $n\alpha$ on flat spaces, but not generally. On the other hand, the heat kernel is always defined (on complete M), whereas (1) and (2) may have no (simultaneous) solution. These facts may relate the above–mentioned Hodge operator to the Dirac/number operator studied here.

2. Definitions and results

Let S be a Dirac bundle over M with Dirac operator D—refer to [LM89] definition II.5.2. In particular S is a real or complex vector bundle with a positive definite symmetric (hermitian) metric $\langle\cdot,\cdot\rangle$ on fibers and a connection ∇. Let $\Gamma(S)$ be the smooth sections of S.

Let $h \in C^\infty(M)$ be a real function and consider the measure $e^{2h}\,dx$. Let $(\phi,\psi)_h = \int_M \langle\phi,\psi\rangle\, e^{2h}\,dx$. Let $L^2_h S$ be the Hilbert space of measurable sections ϕ of S such that $(\phi,\phi)_h < \infty$.

Definition. Let $A = \frac{1}{\sqrt{2}}D$ and $A^* = \frac{1}{\sqrt{2}}D^*$, where D^* is the formal adjoint computed with respect to $(\cdot,\cdot)_h$. Note D is not symmetric (formally self–adjoint) for general h.

Definition. We say that a nonzero $\zeta \in \Gamma(S)$ is a *vacuum state* if $D\zeta = 0$ and $\nabla_{\nabla h}\zeta = 0$.

The first condition on the vacuum is automatic if the label “annihilation operator” is to apply to D. The second condition appeared to the author as a technical condition, but may be a “polarization” in the language of geometric quantization. Clearly if $f \in \Gamma(S)$ is a constant *function* then f is a vacuum.

There is no uniqueness of the vacuum—for instance, all bounded harmonic differential forms on a flat (Riemannian) cylinder are vacuum states for $D = d + \delta$ by this definition, assuming h is a function of the unbounded coordinate.

Fixing a vacuum ζ, we denote the excited states $\varphi_k = (A^*)^k\zeta$ for $k \geq 0$. That is, they are built be applying the creation operator A^* in the usual manner.

Our precise statement of the question is then: Under what conditions on the data (M, g, h, S, D) is the commutator $[A, A^*]$ a positive constant on the linear span of the excited states $\{\varphi_k\}_{k=0}^\infty$ where ζ is a vacuum state?

Our answer is that M must be noncompact, in which case there are necessary and sufficient conditions in the form of partial differential equations on h, that is, on the measure density. Specifically, let α be a positive constant. Then $[A, A^*] = \alpha$ on the span of $\{\varphi_k\}$ if and only if there exists $\gamma \in \mathbf{R}$ such that

$$\triangle h = \alpha \quad \text{and} \tag{1}$$

$$\alpha h + \frac{1}{2}|\nabla h|^2 = \gamma. \tag{2}$$

Here $\triangle = -\mathrm{div} \circ \nabla$ is the nonnegative Laplace–Beltrami operator. Theorem 6 shows that these equations are necessary and theorem 8 that they are sufficient.

Note that these conditions are completely independent of the nature of the particular Dirac bundle S. However, general properties of Dirac operators as a differential operators are essential. (See the calculations in the next section.)

Equations (1) and (2) evidently “overdetermine” h. That is, existence of a solution constrains the Riemannian manifold M. We can give these constraints in more a geometrical form as follows. Let $r = \sqrt{\frac{2}{\alpha}\left(\frac{\gamma}{\alpha} - h\right)}$. (Note (2) implies $h \leq \frac{\gamma}{\alpha}$.) Then equations (1) and (2) are equivalent to

$$\triangle r = 0 \quad \text{and} \tag{3}$$

$$|\nabla r| = 1 \tag{4}$$

respectively, at every point $x \in M$ such that $r(x) > 0$. That is, (1) and (2) are nearly (except where r has value zero and is not differentiable) equivalent to the existence of a nonnegative harmonic distance function.

If r is harmonic then flow along ∇r is volume–preserving.

The substitution $h = h' + \frac{\gamma}{\alpha}$ shows equation (2) has a solution if and only if $\alpha h' + \frac{1}{2}|\nabla h'|^2 = 0$. That is, we can choose $\gamma = 0$ by a normalization of the measure $e^{2h}\, dx$.

Note that equation (1) does not have a solution for $\alpha > 0$ if M is compact (without boundary). For M compact–with–boundary, $\alpha \operatorname{vol} M = \int_M \triangle h\, dx$ is the integral of an $(n-1)$–form on ∂M by the divergence theorem. Specifically, if M is complete (as a manifold–without–boundary) and the hypotheses of theorem 6 apply then M is not compact.

3. Necessity of equations (1) and (2)

If $\{e_j\}$ is a local orthonormal basis of $\Gamma(TM)$ then the Dirac operator for S is $D = e_j \cdot \nabla_{e_j}$ where "$\cdot$" is Clifford multiplication $\Gamma(TM \otimes S) \to \Gamma(S)$. (The Einstein convention is used throughout.) Since (M, g) is Riemannian, Clifford multiplication is nondegenerate: $\langle X \cdot \phi, X \cdot \phi \rangle = -g(X, X)\, \langle \phi, \phi \rangle$ for (real) $X \in \Gamma(TM)$ and $\phi \in \Gamma(S)$. If f is a function and if ϕ is a section of S then $D(f\phi) = \nabla f \cdot \phi + f D\phi$.

Recall that we have both the Levi–Civita connection ∇ on TM and a connection (again ∇) on S. Their relation is $\nabla_X (Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X \phi$, where $X, Y \in \Gamma(TM)$, $\phi \in \Gamma(S)$.

Recall $\triangle = \delta d = -\operatorname{div} \nabla = -\operatorname{tr} \operatorname{Hess}$.

The following second order product rules undoubtedly appear somewhere in the literature of Dirac operators. They have elementary proofs.

Lemma 1. *For any smooth function f on M, $D(\nabla f) = \triangle f$.*

Lemma 2. *For $f \in C^\infty(M)$ and $\phi \in \Gamma(S)$,*

$$D\left(\nabla f \cdot \phi\right) + \nabla f \cdot D\phi = (\triangle f)\phi - 2\nabla_{\nabla f}\phi. \tag{5}$$

Corollary 3. $D^2(f\phi) = (\triangle f)\phi - 2\nabla_{\nabla f}\phi + f D^2\phi$.

We can now express A^* and $[A, A^*]$ as differential operators, using the usual integration by parts.

Corollary 4. *On $\phi \in \Gamma(S)$,*

$$A^*\phi = \frac{1}{\sqrt{2}}\left(D\phi + 2\nabla h \cdot \phi\right),$$

$$[A, A^*]\,\phi = -2\left(\nabla h \cdot D\phi + \nabla_{\nabla h}\phi\right) + (\triangle h)\phi.$$

We now show the necessity of equations (1) and (2). Regarding vacuum states, note that the usual elliptic theory implies that ζ is smooth if $D\zeta = 0$. Also, if $\zeta \neq 0$ and $D\zeta = 0$ then $\{x \in M : \zeta(x) = 0\}$ has empty interior and measure zero.

Given a vacuum state ζ, recall $\varphi_k = (A^*)^k\zeta$ for $k = 0, 1, 2, \dots$. Let $\mathcal{D}^\zeta = \operatorname{span}\{\varphi_k\}$ be the vector space of finite linear combinations.

It is useful to expose the space $\mathcal{P}_h$ on which the commutator $[A, A^*]$ acts in a simple manner. (We will see that if (1) and (2) then the excited states φ_k are in $\mathcal{P}_h$ and indeed the commutator acts as a constant on these states.)

Definition. Let $\Pi_h = \nabla h \cdot D + \nabla_{\nabla h}$. Let $\mathcal{P}_h = \ker \Pi_h \subset \Gamma(S)$.

By corollary 4, the commutator $[A, A^*]$ acting on $\mathcal{P}_h$ is multiplication by the scalar $\triangle h$.

Lemma 5.

$$\Pi_h(h\phi) = h\Pi_h\phi$$

$$\Pi_h(\nabla h \cdot \phi) = |\nabla h|^2 D\phi - \nabla h \cdot \nabla_{\nabla h}\phi + (\triangle h)\nabla h \cdot \phi + \nabla\left(\frac{1}{2}|\nabla h|^2\right) \cdot \phi.$$

Proof. The first formula corresponds to a straightforward calculation.

From lemma 2 and corollary 3, $\Pi_h(\nabla h\cdot\phi) = |\nabla h|^2 D\phi + (\triangle h)\nabla h\cdot\phi - \nabla h\cdot(\nabla_{\nabla h}\phi) + \frac{1}{2}\left((\triangle h)\nabla h - D^2(h\nabla h) + hD^2(\nabla h)\right)\cdot\phi = |\nabla h|^2 D\phi - \nabla h \cdot \nabla_{\nabla h}\phi + \frac{3}{2}(\triangle h)\nabla h \cdot \phi - \frac{1}{2}D\left(\nabla h \cdot \nabla h + hD(\nabla h)\right)\cdot\phi + \frac{1}{2}hD(\triangle h)\cdot\phi$ which gives the second result by lemma 1. □

Theorem 6. *Suppose $\zeta \in \Gamma(S)$ is a vacuum state. If $[A, A^*] = \alpha$ (a positive constant) on $\mathcal{D}^\zeta$ then A and $N = A^*A$ leave $\mathcal{D}^\zeta$ invariant and there exists a $\gamma \in \mathbf{R}$ so that equations* (1) *and* (2) *hold for h.*

Proof. It is clearly the case that A, N leave $\mathcal{D}^\zeta$ invariant, by construction and the assumed commutation relation.

To show equation (1), note that $\alpha\zeta = [A, A^*]\,\zeta = -2\left(\nabla h \cdot D + \nabla_{\nabla h}\right)\zeta + (\triangle h)\zeta = (\triangle h)\zeta$, and thus $\triangle h = \alpha$ where $\zeta \neq 0$, and thus on all of M.

To show equation (2) first compute that $A^*\zeta = \sqrt{2}\nabla h \cdot \zeta$. By equation (5), corollary 4, and the hypotheses, $\alpha\nabla h \cdot \zeta = -2\left(\nabla h \cdot D + \nabla_{\nabla h}\right)(\nabla h \cdot \zeta) + \alpha\nabla h \cdot \zeta$, so $\Pi_h(\nabla h \cdot \zeta) = 0$. On the other hand, by lemma 5, the definition of a vacuum, and equation (1), $\Pi_h(\nabla h \cdot \zeta) = \alpha h \cdot \zeta + \left(\frac{1}{2}|\nabla h|^2\right) \cdot \zeta = \nabla\left(\frac{1}{2}|\nabla h|^2 + \alpha h\right) \cdot \zeta$.

Thus $\nabla\left(\frac{1}{2}|\nabla h|^2 + \alpha h\right)\cdot\zeta = 0$. If $x \in M$ then either $\zeta(x) = 0$ or $\nabla(\alpha h + \frac{1}{2}|\nabla h|^2) = 0$ by the nondegeneracy of Clifford multiplication. From preceding remarks on vacuums, equation (2) holds everywhere. □

4. Sufficiency of equations (1) and (2)

Lemma 7. *Suppose $h \in C^\infty(M)$ satisfies equations* (1) *and* (2) *for some $\alpha > 0$ and suppose ζ is a vacuum. Then $\varphi_k \in \mathcal{P}_h$ for all k and*

$$\varphi_k = \begin{cases} \sum_{i=0}^{j} a_{ki}h^i\zeta, & k = 2j \text{ is even,} \\ \sum_{i=0}^{j} b_{ki}h^i\nabla h \cdot \zeta, & k = 2j+1 \text{ is odd,} \end{cases} \tag{6}$$

where a_{ki}, b_{ki} are real constants depending only on α and γ.

Proof. Note that since ζ is a vacuum, for all $j = 0, 1, 2, \ldots$ we have

$$A^*(h^j\zeta) = \frac{1}{\sqrt{2}}(jh^{j-1} + 2h^j)\nabla h \cdot \zeta$$
$$A^*(h^j\nabla h \cdot \zeta) = -\frac{1}{\sqrt{2}}(jh^{j-1} + 2h^j)|\nabla h|^2\zeta + \frac{1}{\sqrt{2}}h^j(\triangle h)\zeta.$$

Thus equations (1) and (2) imply equation (6) by induction. Then lemma 5 implies $\varphi_k \in \mathcal{P}_h$. □

Theorem 8. *Suppose $h \in C^\infty(M)$ satisfies equations* (1) *and* (2) *for some $\alpha > 0$ and suppose ζ is a vacuum. Then $[A, A^*] = \alpha$ on $\mathcal{D}^\zeta$ and $N\varphi_k = k\alpha\varphi_k$.*

Proof. Use the above lemma and the usual induction. □

It is time to admit that everything so far has been "formal" in the sense that A, A^*, etc. act in the space of smooth sections of S, and that the Hilbert space L^2_hS has played no role beyond integration–by–parts.

Let us show the existence of a Hamiltonian (Schrödinger) operator corresponding to N. In fact, if equations (1) and (2) hold then we find (as follows) an essentially self–adjoint "extension" $\tilde{N}$ of N. This $\tilde{N}$ has core $\Gamma_c(S)$ and is unitarily–equivalent to an operator of the form $D^2 + V$, for scalar V, under the transform to the unweighted L^2 space of sections of S. It is an extension in the sense that if $\phi \in \mathcal{P}_h$ then $N\phi = \tilde{N}\phi$.

Definition. Let $\tilde{D} = D + \nabla h\cdot$, let $\tilde{\triangle} = \tilde{D}^2$, let $V_h = -\alpha h - \frac{\alpha}{2} + \gamma$, and let

$$\tilde{N} = \frac{1}{2}\tilde{\triangle} + V_h.$$

These operators are symmetric in L^2_hS with dense domain $\Gamma_c(S)$.

Lemma 9. *If (M, g) is complete then $\tilde{D}$ and $\tilde{\triangle}$ are essentially self–adjoint in L^2_hS.*

Proof. Let L^2S be the space of measurable sections of S which are square–integrable with respect to dx. Let $U : L^2_hS \to L^2S$ be the unitary map ("ground state transform") $\phi \mapsto e^h\phi$. Then $\tilde{D}$, $\tilde{\triangle}$ are unitarily equivalent to D, D^2 in L^2S (with dense domains $\Gamma_c(S)$), respectively. Clearly $\Gamma_c(S)$ is preserved by U. It is well–known that D, D^2 are self–adjoint if M is complete. □

Theorem 10. *Suppose (M, g) is complete. If h solves equation* (2) *for $\alpha > 0$ and $\gamma \in \mathbf{R}$, then V_h is bounded below and thus $\tilde{N}$ is essentially self–adjoint in L^2_hS. If in addition h solves equation* (1) *then $\phi \in \mathcal{P}_h$ implies $N\phi = \tilde{N}\phi$.*

Proof. Note $U\tilde{N}U^{-1} = \frac{1}{2}D^2 + V_h$ is a self–adjoint operator plus a bounded–below scalar potential, which acts in unweighted L^2S. Theorem 2.3 of [Les00] shows $\frac{1}{2}D^2 + V_h$ is self–adjoint.

For $\phi \in \mathcal{P}_h$ it follows that $\nabla h \cdot D\phi = -\nabla_{\nabla h}\phi$. Thus from lemma 2

$$\begin{aligned} N\phi &= \frac{1}{2}D^2\phi + \nabla h \cdot D\phi = \frac{1}{2}D^2\phi - \nabla_{\nabla h}\phi \\ &= \frac{1}{2}\big(D^2\phi + D(\nabla h \cdot \phi) + \nabla h \cdot D\phi - (\triangle h)\phi\big) \\ &= \frac{1}{2}\big(\tilde{D}^2\phi - \nabla h \cdot \nabla h \cdot \phi - (\triangle h)\phi\big) \\ &= \frac{1}{2}\tilde{\triangle} + \frac{1}{2}|\nabla h|^2\phi - \frac{1}{2}(\triangle h)\phi. \end{aligned}$$

Thus $N\phi = \tilde{N}\phi$ if equations (1) and (2) hold for h and if $\phi \in \mathcal{P}_h$. □

In particular we conclude that under (1) and (2), $\tilde{N}\varphi_k = k\alpha\varphi_k$. Of course this does not yet imply that $\{k\alpha\}_{k=0}^{\infty}$ lies in the spectrum of $\tilde{N}$. But we can show that the excited states φ_k are integrable under a geometric condition. We then get the spectral conclusion by an easy argument.

The example where S is the bundle of forms on $\mathbf{R}^2$, $D = d + \delta$, and $h = -(x_1)^2$ shows that some further condition is necessary for integrability of the excited states.

Assume (2) holds for $h \in C^\infty(M)$ and that $\gamma = 0$. Let $r = \sqrt{-\frac{2}{\alpha}h}$ and note $|\nabla r| = 1$. Suppose (without loss of generality) that $r(x_0) = 0$ and fix $x_0 \in M$. Let $M_s = r^{-1}(s)$ for $s > 0$. By (2) and the implicit function theorem, M_s is an $(n-1)$–dimensional submanifold of M.

Lemma 11. *Assume M is complete and that $h \in C^\infty(M)$ satisfies (1) and (2) for some $\alpha > 0$. Suppose that for the vacuum ζ there exists $0 \le c < \frac{\alpha}{2}$ and $C_0 > 0$ such that $|\zeta(x)| \le C_0 e^{cs^2}$ for $x \in M_s$. Furthermore, suppose $\mathrm{vol}_{n-1} M_s < \infty$ for some $s > 0$. Then $\varphi_k \in L^2_h S$ for any $k \ge 0$.*

Proof. By (1), $\mathrm{vol}_{n-1} M_s$ is constant independent of s and also $r^{-1}(0)$ is a set of measure zero. Thus

$$\int |h|^k|\zeta|^2 e^{2h}\,dx = (\mathrm{vol}_{n-1} M_s)\left(\frac{\alpha}{2}\right)^k \int_{s>0} s^{2k} e^{cs^2} e^{-\alpha s^2}\,ds < \infty.$$

By lemma 7, $\varphi_k \in L^2_h S$ for k even. And then equation (2) gives the result for k odd. □

Corollary 12. *Under the hypotheses of lemma 11 above, φ_k is in the domain of $\tilde{N}$ (as a self–adjoint operator) and $k\alpha$ is in the spectrum of $\tilde{N}$ for all $k \ge 0$.*

Proof. We need only show that φ_k is in the domain of the closed operator $\tilde{N}$, and this can be done directly using the usual smooth–cutoff argument. Note first that if $f = f(r)$ and $\phi \in \mathcal{P}_h$ then $f\phi \in \mathcal{P}_h$. Thus $\tilde{N}(f\varphi_k) = N(f\varphi_k)$, and the later can be expanded by the various formulas given in the second section. Also note that $\nabla_{\nabla f}\varphi_k = -\nabla f \cdot D\varphi_k = -\sqrt{2}\nabla f \cdot A\varphi_k$ if $f = f(r)$. □

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Dept of Mathematical Sciences, University of Alaska, Fairbanks AK 99775
E-mail address: ffelb@uaf.edu