# *Collocation approximation of the monodromy operator of periodic, linear DDEs*

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# Thanks

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# Outline

1. stability chart examples from machining
2. abstractions for periodic, linear DDEs
3. Chebyshev collocation
4. estimates for ODE initial value problems
5. approximation of the monodromy operator

# Example 1: Turning

1 DOF model for regenerative vibrations of cutting tool with mass $m$, stiffness $k$, and damping $c$:

$$m\ddot{x} + c\dot{x} + kx = \Delta F_x$$

$\Delta F_x = \Delta F_x(f)$ is $x$-component of cutting force variation, fcn of chip thickness $f$. Linearizing at prescribed thickness $f_0$ gives (for $k_1$ is constant)[a]

$$\Delta F_x \approx k_1 \left( x(t-\tau) - x(t) \right)$$

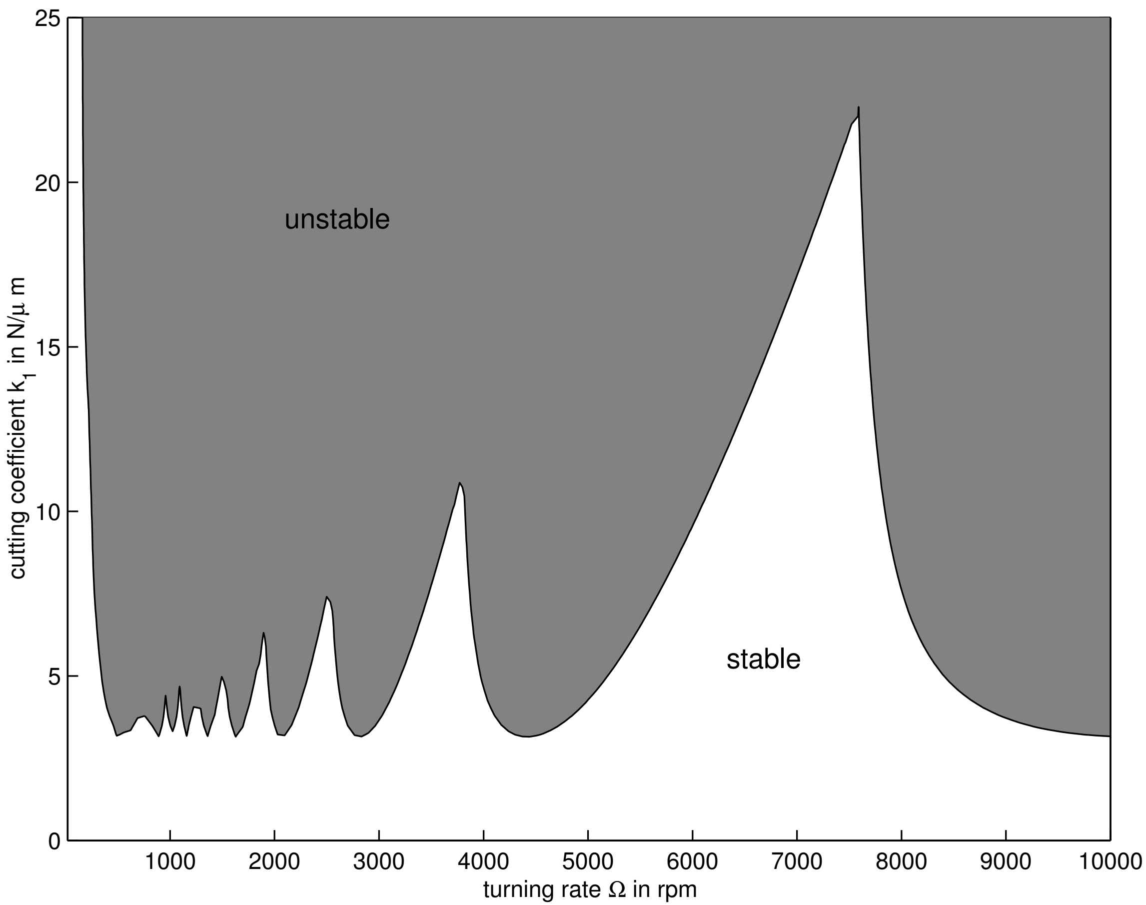
[a] $\tau$ is rotation time of workpiece; $\Omega = 60/\tau$ is rot. rate (RPM)

# Example 1: Turning, cont.

*QUESTION*: Suppose $m, c, k$ are fixed. For which values $\Omega$, $k_1$ is this turning DDE (linearly) stable[a]?

[a] *Definition*. A linear, homogeneous DDE is *stable* (i.e. asymptotically stable) if all solutions decay to zero.

# Example 1: Turning stability chart



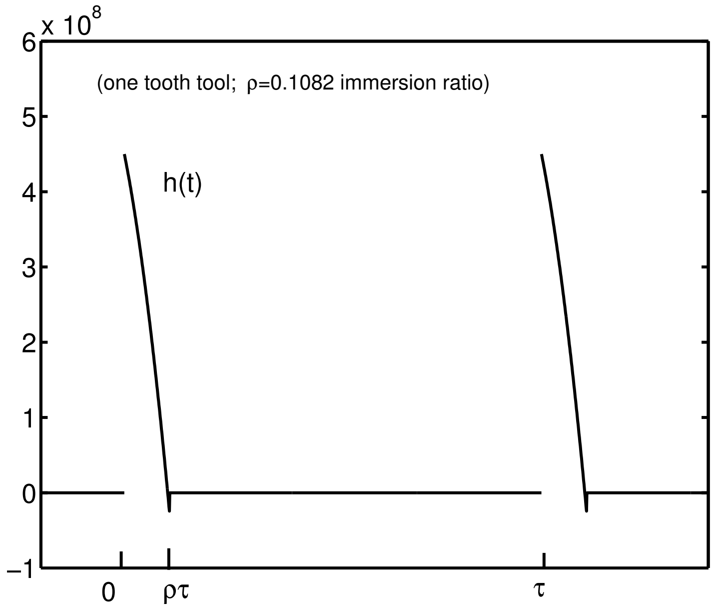

(Based on $150 \times 150$ points in parameter plane. Compare to exact chart. For $\Omega \gtrsim 1000$, boundary comes within one point of correct. For $\Omega \lesssim 1000$, problem is stiffness, below.)

# Example 2: (Interrupted) milling

1 DOF linearized model for regenerative vibrations:

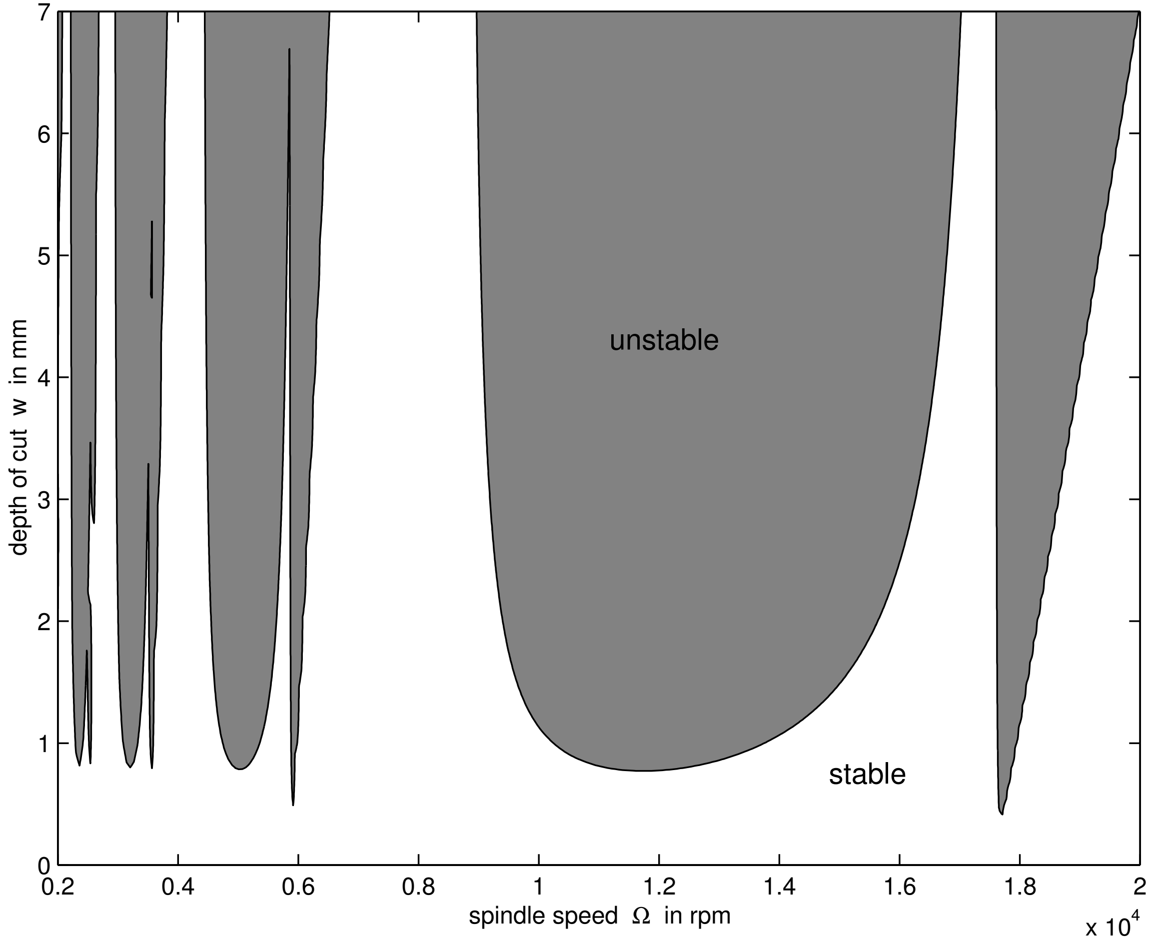
$$m\ddot{x} + c\dot{x} + kx = wh(t)(x(t-\tau) - x(t))$$

*But* $h(t)$ has the following nonsmooth, time-dependent form:




*QUESTION*: Suppose $m, c, k$ all fixed. For which values $\Omega = 60/\tau$, $w$ is this milling DDE stable?

# Example 2: Milling stability chart




(Compare to Insperger, et al., *Multiple chatter frequencies in milling processes*, J. Sound Vibration (2003).

# Conventions

- We consider *linear, periodic-coefficient DDEs with fixed delays*. We assume rational relations among delays and coefficient periods. (For this talk: only one delay and period=delay.)
- Put in standard first-order form
$$\dot{\mathbf{y}}(t) = A(t,\epsilon)\mathbf{y}(t) + B(t,\epsilon)\mathbf{y}(t-\tau)$$
where $A, B$ have $\tau$-periodic dependence on $t$ and depend continuously on parameters $\epsilon \in \mathbb{R}^d$ (typically $d = 1, 2, 3$).
- We assume $A, B$ are *piecewise analytic* functions of $t$.

# Our Mission

- Construct a fast and accurate numerical method (based on *Chebyshev collocation*, below) for stability charts for linear, periodic DDE problems with piecewise-analytic coefficients.
- Prove it works. (Prove estimates for accuracy of IVP solutions. Prove estimates for eigenvalues.)
- Build an easy to use MATLAB package to implement it.

(STATUS July 2004: Mostly done including estimates (for constant non-delayed-coefficient cases). MATLAB suite in early version. See web site `www.cs.uaf.edu/~bueler/DDEcharts.htm`.)

# Recall (for linear, periodic DDE)

Initial value problem

$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{y}_{-\tau}, \quad \mathbf{y}(t) = \phi(t) \text{ for } t \in [-\tau, 0]$$

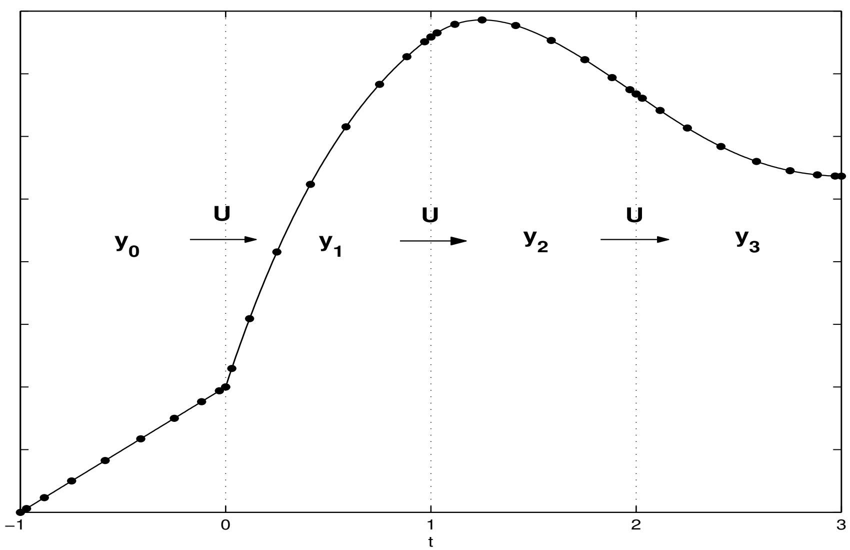
has solution (monodromy; delayed FTM):

$$(U\mathbf{f})(t) = \Phi(t)\left[\mathbf{f}(1) + \int_{-1}^{t} \Phi^{-1}(s)B(s)\mathbf{f}(s)\,ds\right]$$

(where $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$).

Soln of IVP:

$$\mathbf{y}_{n+1} = U\mathbf{y}_n,\ \mathbf{y}_0 = \phi$$



# Abstract view of linear, periodic DDE

$U$ is a *compact*[a] operator on $C([0,\tau])$.

Our class of DDE are simply linear difference eqns with compact generator in $C([0,\tau])$: $\mathbf{y}_{n+1} = U\mathbf{y}_n$.

Compact ops are (norm-)limits of finite rank operators.

Stability: $\rho(U) < 1$ if and only if DDE is stable.[b]

[a]It is formed from an integral operator and $\mathbf{f} \mapsto \mathbf{f}(1)$, a finite rank operator.

[b]Caveat: this is *eigenvalue* stability. Degree of nonnormality of $U$ does matter.

# Chebyshev poly approx: 3 good reasons

- Polynomial and Fourier approximation ("spectral approximation") converges faster than finite diff or finite elem or cubic splines or wavelets on analytic functions.
- Though the coefficients in our DDE are periodic *the solutions are not*. Thus Fourier not so good. (Also: poly approx can be good on each piece of a piecewise-analytic fcn without generating Gibbs phenomena.)
- Chebyshev points are nearly optimal polynomial interpolation points for minimizing uniform error.

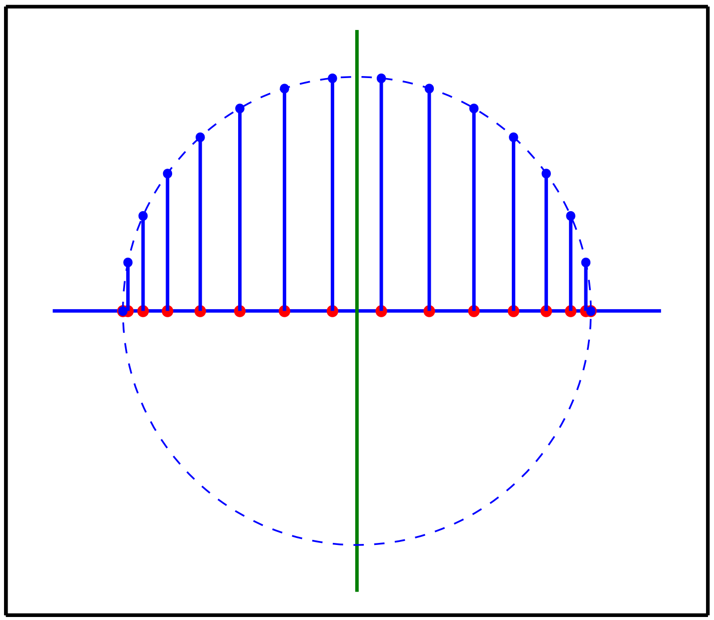
# Chebyshev collocation points

Chebyshev poly approx can be implemented by *collocation*.

For degree $N$, Cheb collocation points are

$$t_j = \cos(j\pi/N), \qquad j = 0, \ldots, N.$$

($t_j$ are projections of equally-spaced points on unit circle[a]).



Note $t_j \in [-1, 1]$. (If needed, shift the $t_j$ to interval $[0, \tau]$.)

[a]The *Fourier* collocation points. Cheb collocation can be implemented by FFT.

# Cheb spectral differentiation

1. Given $f(t)$ on $[-1, 1]$.
2. Construct interpolating polynomial $p(t)$: $p(t_j) = f(t_j)$.
3. Find $\dot{p}$.
4. Evaluate it at $t_j$: $\dot{f}(t_j) \approx \dot{p}(t_j)$.

This gives a matrix approximation of derivative $\frac{d}{dt}$:

$$\dot{f} \qquad \approx \qquad \dot{p} \qquad \overset{\text{(represented by)}}{=} \qquad D_N v$$

# Cheb collocation approx of $U$

Use Cheb matrix approximations: (*i*) $D_N \approx \frac{d}{dt}$ (of a vector-valued fcn); (*ii*) $M_A \approx$ (mult by $A(t)$); (*iii*) $M_B \approx$ (mult by $B(t)$). Modify these to incorporate ODE initial condition: $\mathbf{y}(0) = \phi(0)$.

$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{y}_{-\tau} \text{ with } \mathbf{y}(t) = \phi(t), t \in [-\tau, 0]$$

is approximated by

$$D_N \mathbf{v} = M_A \mathbf{v} + M_B \mathbf{w}$$

(here $\mathbf{v} \approx \mathbf{y}$, $\mathbf{w} \approx \phi$).

Solving for $\mathbf{v}$ *is* approximating $U$:

$$U \approx U_N \equiv (D_N - M_A)^{-1} M_B.$$

# Example: Scalar DDE

Consider scalar DDE: $\dot{x} = -x + (1/2)x_{-2}$ with $N = 3$. Then $D_N$, $M_A$, $M_B$, and $U_N = (D_N - M_A)^{-1} M_B$ are $4 \times 4$ matrices. Last rows modified to enforce initial condition. $D_N$, $U_N$ generally dense.

$$M_A = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \quad M_B = \begin{pmatrix} 1/2 & & & \\ & 1/2 & & \\ & & 1/2 & \\ 1 & & & 0 \end{pmatrix}$$

$$D_N = \begin{pmatrix} 19/6 & -4 & 4/3 & -1/2 \\ 1 & -1/3 & -1 & 1/3 \\ -1/3 & 1 & 1/3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_N = \begin{pmatrix} 0.2058 & 0.2469 & 0.1152 & 0 \\ 0.1852 & 0.2222 & 0.2037 & 0 \\ 0.6626 & -0.1049 & 0.2510 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

# Example: Eigs of a scalar DDE

For $\dot{x} = -x + (1/2)x_{-1}$ we compare $U_N$ eigenvalues to exact:

"Exact" *method*: Each root $\mu \in \mathbb{C}$ of characteristic eqn $\mu = -1 + 0.5e^{-\mu}$ is eigenvalue of $U$. Reduce char eqn to real variable problem. Solve by robust one-variable method (e.g. bisection) to $10^{-14}$ relative accuracy.

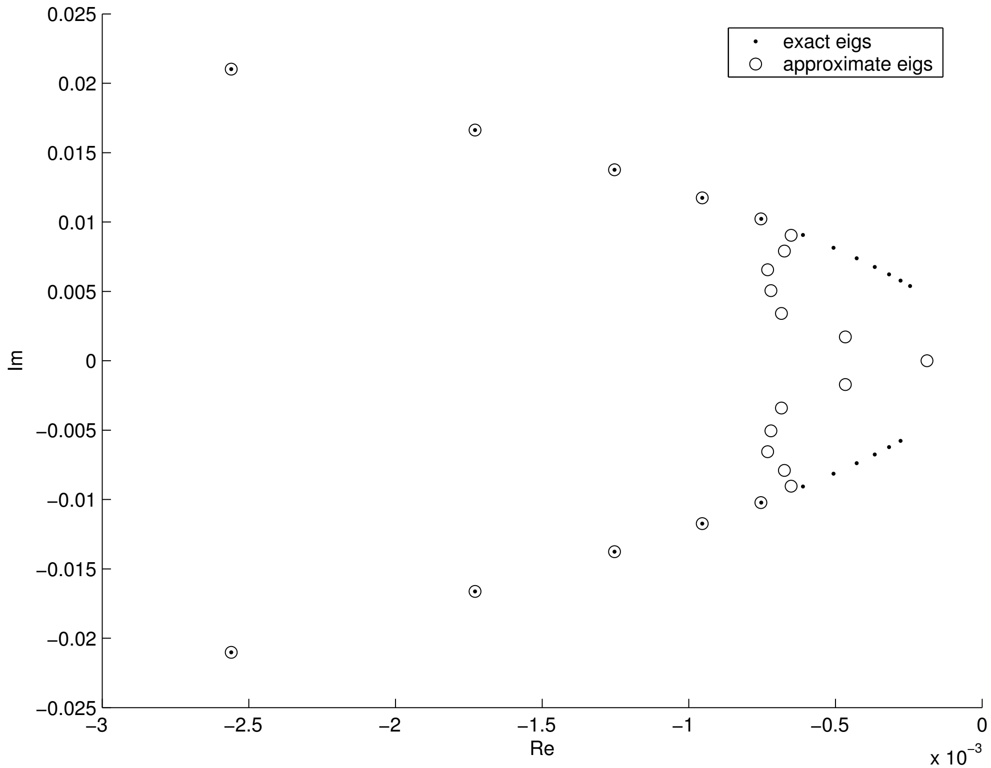
vs

*Cheb collocation* with $N = 29$: Compute $U_N$. Find eigs of $U_N$.

RESULT: Largest 7 eigenvalues of $U_N$ are each accurate to more than 12 digits.

# Example, cont

For remaining 23 eigenvalues, here's the picture:




SUMMARY: Over 100 digits of correct eigenvalues from $30 \times 30$ matrix approx of $U$.
Only eigs near $0 \in \mathbb{C}$ are inaccurate (irrelevant for stability).

# Cost of a stability chart

Using numerical method to produce $m \times m$ approximation to $U$, the time to produce a chart is

$$O((\text{\# of pixels}) \cdot m^3)$$

with standard estimates on QR method for eigenvalues.

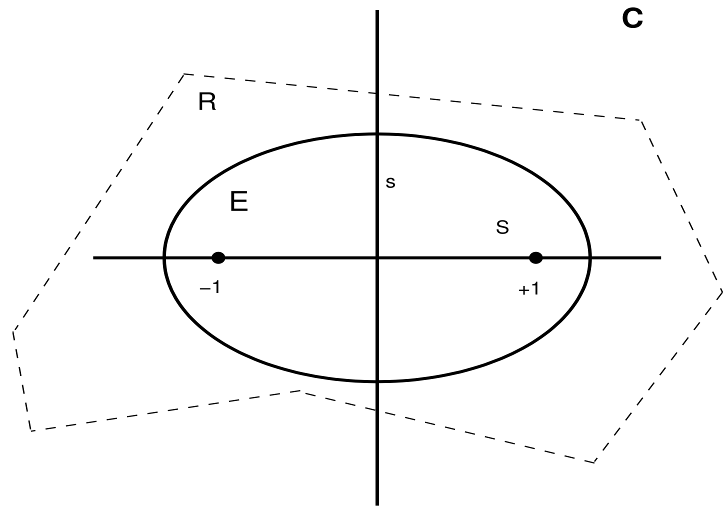
$m$ matters! Small is good!

# Accuracy of Chebyshev interpolation

*Theorem* [classical]. Let $p$ be degree $N$ poly for $f$ using $N+1$ Cheb colloc pts. If $f$ analytic in a $\mathbb{C}$-neighborhood $R$ of $[-1,1]$ then there exists $C$ s. t.

$$\|f-p\|_\infty \le C(S+s)^{-N}$$

where $S, s$ are semi axes of ellipse $E$ s. t. $[-1,1] \subset E \subset R$.




Moral: If $f$ analytic then $p$ improves by a fixed number of digits per increase by one in $N$.

# Accuracy of DDE collocation soln

*Theorem*. Consider IVP $\dot{y} = ay + b(t)y_{-\tau}$, $y(t) = \phi(t)$ for $t \in [-\tau, 0]$. Let $q$ be the interpolating poly of delayed term $b\phi$. Find degree $N$ collocation solution $p(t)$, a polynomial. Then

$$\|y - p\|_\infty \leq c_1 \|q - b\phi\|_\infty + c_2 |\dot{p}(0) - a\phi(0) - b(0)\phi(-\tau)|.$$

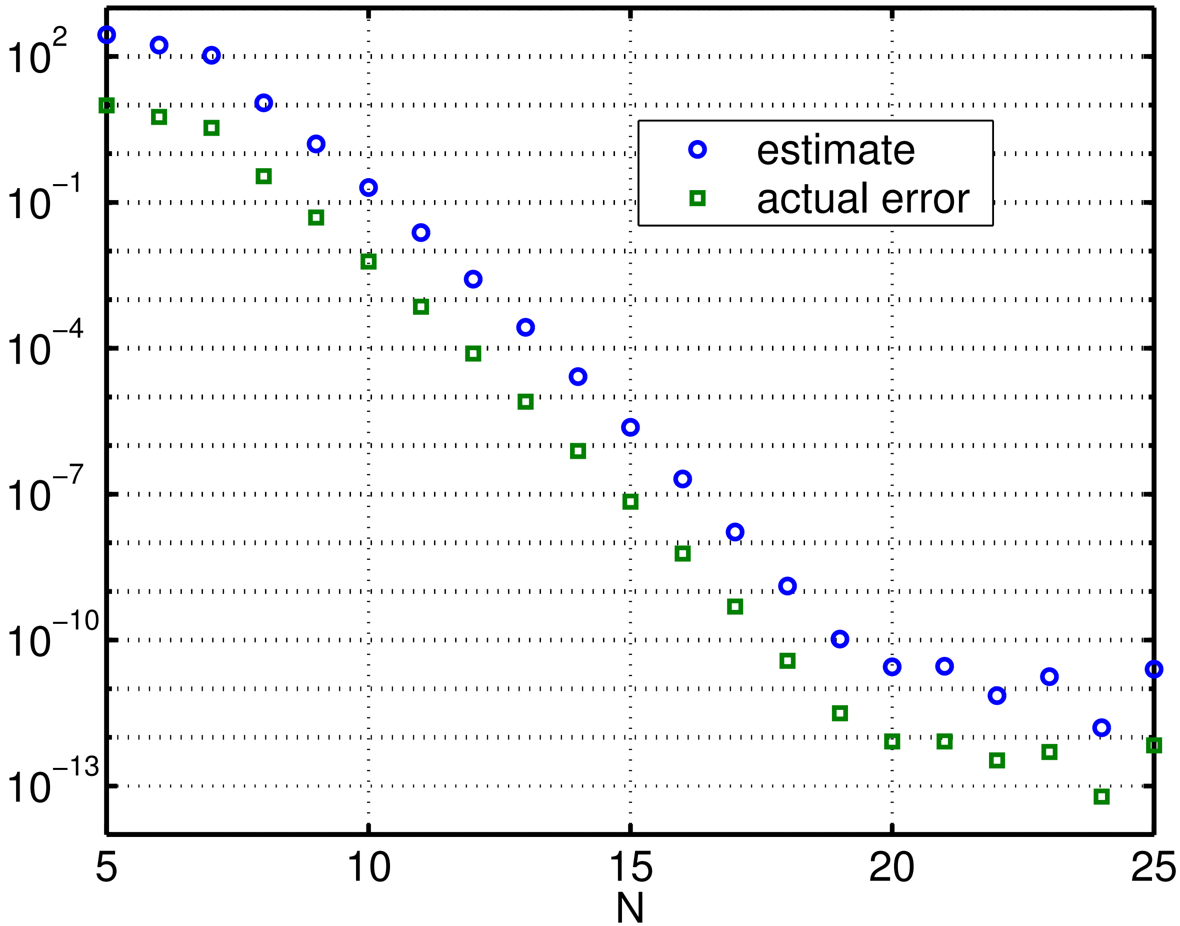
$c_1, c_2$ depend on $a$ but are $O(1)$ in $N$.

Thus

- Error has two sources: (i) interpolation error for delayed term; (*ii*) residual error at initial time from difficulty of nonhomogeneous ODE problem.
- *a posteriori* result: Do computation, get *proven* estimate of quality of solution based on result.

# Example: accuracy in DDE IVP

Find $y(t)$ on $[0, 2]$ if $\quad \dot{y} = 3y + (t-1)y_{-2}, \quad \phi(t) = 1.$



# Estimates for eigenvalues of $U$

In basis of Chebyshev polynomials $\{T_j\}$, matrix entries of $U$ on $C([-1,1])$ can be computed by inner products: $U_{jk} = \langle T_j, UT_k \rangle$.

Note $y = UT_k$ is the solution of an IVP. We use previous *a posteriori* estimate to show $\|UT_k - (U_N)T_k\|$ small[a] for $k$ up to about $\frac{3}{4}N$.

Now use eigenvalue perturbation theory[b] to show large eigenvalues of $U_N$ are close to those of $U$.

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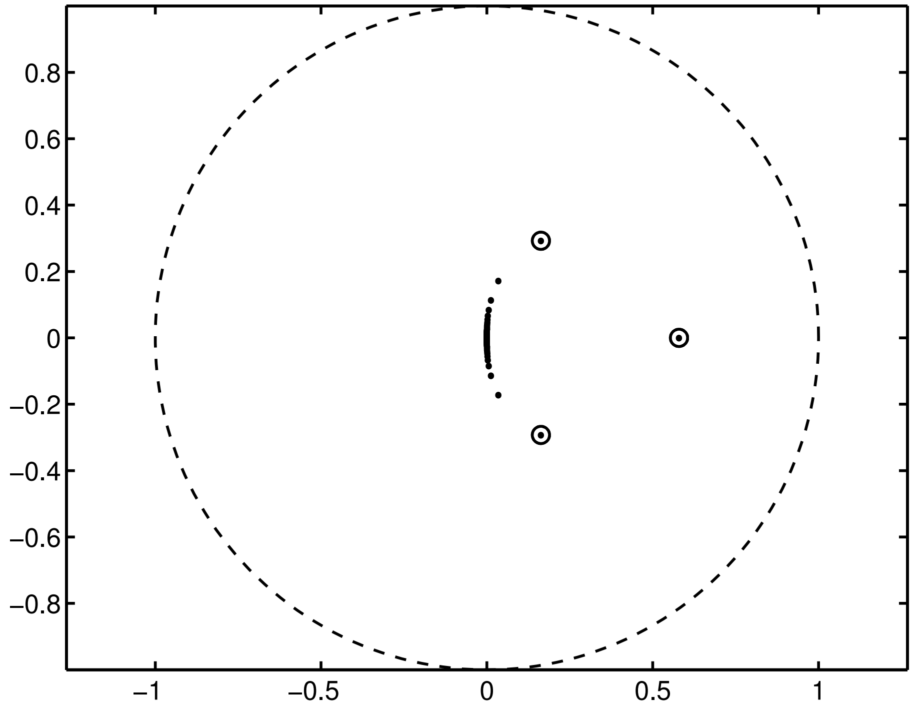
[a] Recall $U_N$ is Chebyshev approximation to $U$.

[b] An extension of the Bauer-Fike theorem to compact operators on Hilbert spaces; need to transfer $U$ to act on Sobolev space $H^1_{Cheb}$.

# Provable eigenvalues of $U$.

**Example:** Consider $\dot{y} = -2y + (1 + \sin(3\pi t))y_{-2}$. Let $N = 95$.

**Result:** Dots are eigs of $U_N$; discs are *proven* error bounds for sufficiently large eigs of $U$. (If $\mu$ is an eig of $U$ and $|\mu| \geq 0.2$ then $\mu$ is in one of these discs.) This DDE is *proven stable*.




Size of discs drops exponentially with increasing $N \gtrsim 90$ (this example).

# Why one really cares about $U$

The interesting systems are *nonlinear* DDEs. The linear, periodic DDEs are just their linearizations.
Questions about nonlinear DDE:

- find fixed points and periodic orbits
- nature of bifurcations?

To study the latter question we need good
*bases for spaces of stable and unstable directions*
Good approximation to $U$ means good bases for these purposes.

But that's another talk . . .

See web site `www.cs.uaf.edu/~bueler/DDEcharts.htm`.