Reliable stability charts for periodic delay differential equations

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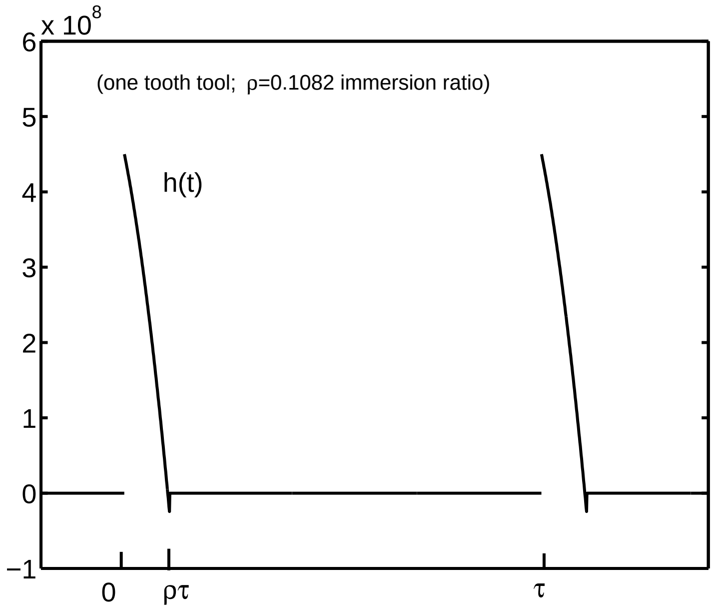
Example: regenerative vibrations in milling

1 DOF linearized model:

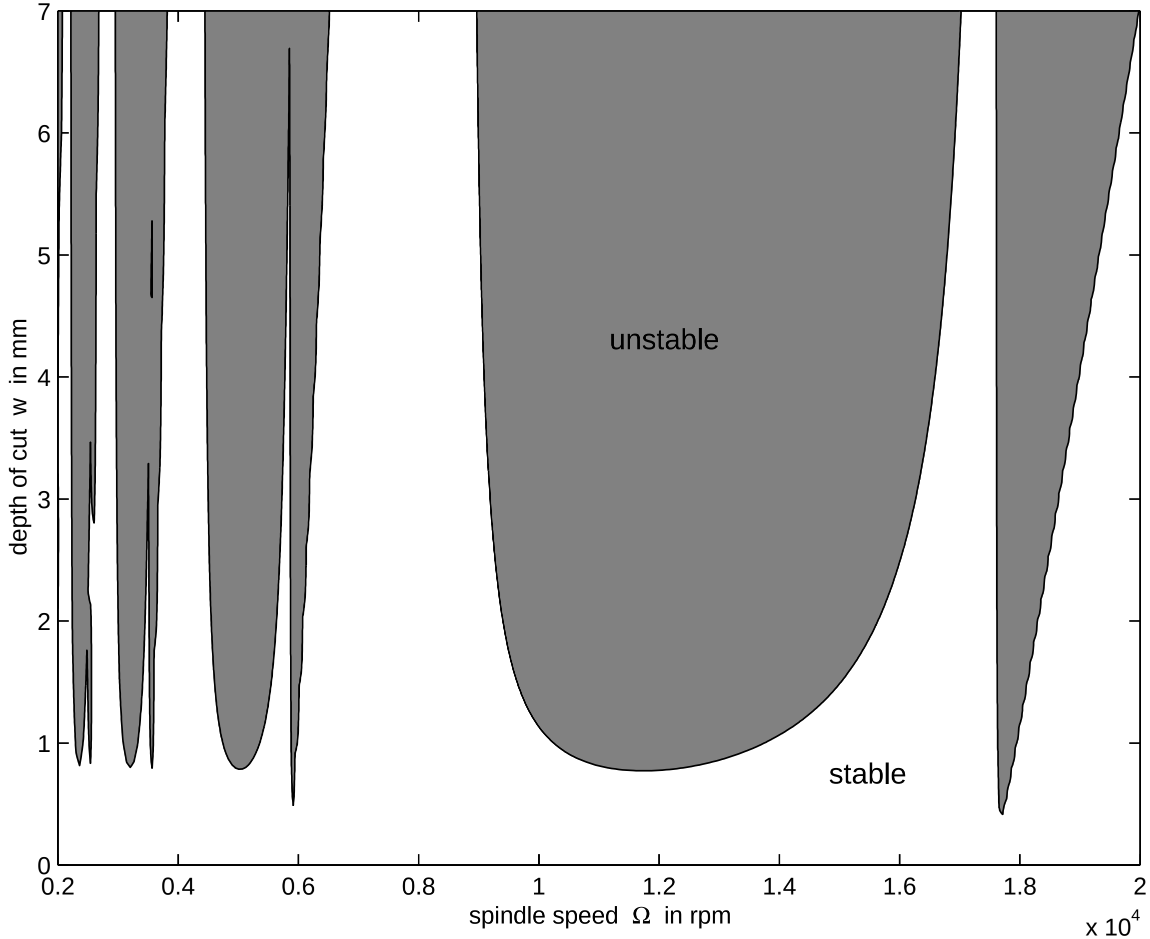
$$m\ddot{x} + c\dot{x} + kx = wh(t)\left[x(t-\tau) - x(t)\right]$$

where $h(t)$ has nonsmooth, time-dependent form:



QUESTION: Suppose m, c, k fixed. For which values of parameters $\Omega = 60/\tau$ and w is this milling DDE stable?

Example, cont.: Milling stability chart



(Compare to Insperger, et al., *Multiple chatter frequencies in milling processes*, J. Sound Vibration (2003).

Conventions

- We consider only *linear, periodic-coefficient DDEs with fixed delays*. For this talk: one delay and period=delay.
- notation: $y_{-\tau}(t) = y(t-\tau)$
- Put in standard first-order system form

$$\dot{y}(t) = A(t,\epsilon)y(t) + B(t,\epsilon)y_{-\tau}(t)$$

where A, B have τ-periodic dependence on t and depend continuously on parameters $\epsilon \in \mathbb{R}^d$.

Our project

We have

- created a fast and accurate numerical method based on *Chebyshev collocation* for solving initial value problems *and* for computing stability charts for linear, periodic DDE problems;
- proven it works by finding *a posteriori* estimates on both the IVPs *and on the computed eigenvalues*;
- and implemented it as an easy to use MATLAB package:

`www.cs.uaf.edu/~bueler/DDEcharts.htm`

Monodromy operator U of DDE

For linear DDE the initial value problem

$$\dot{y} = A(t)y + B(t)y_{-\tau}, \quad y(t) = \phi(t) \text{ for } t \in [-\tau, 0]$$

has solution $y = U\phi$ by a *monodromy operator* U:

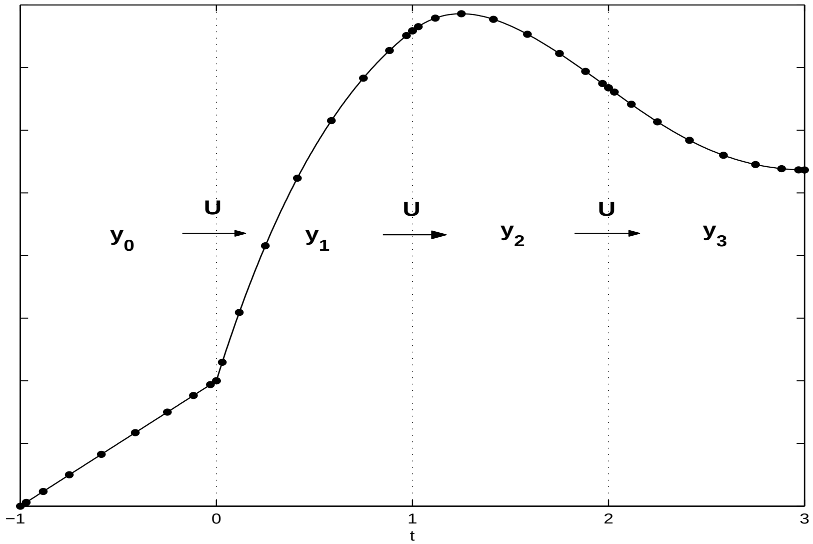
$$(U\phi)(t) = \Phi(t)\left[\phi(0) + \int_0^t \Phi^{-1}(s)B(s)\phi(s-\tau)\,ds\right]$$

where $\Phi(t)$ is fundamental soln of ODE: $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$.

Periodic coefficients $A(t), B(t)$ then implies abstract view:

$$y_{n+1} = Uy_n,$$

$$y_0 = \phi$$



Abstract view of linear, periodic DDE

- U is a *compact* operator on $C([0,\tau])$
- our class of DDE are simply linear difference eqns with compact generator in $C([0,\tau])$: $y_{n+1} = Uy_n$
- compact operators are (norm-)limits of finite rank operators
- stability: $\rho(U) < 1$ if and only if DDE is (asymptotically) stable[a]

Next: numerical approximation of the DDE and of U

[a] Caveat: Eigenvalues determine only ultimate stability. U is typically nonnormal so much solution growth is possible even when $\rho(U) < 1$.

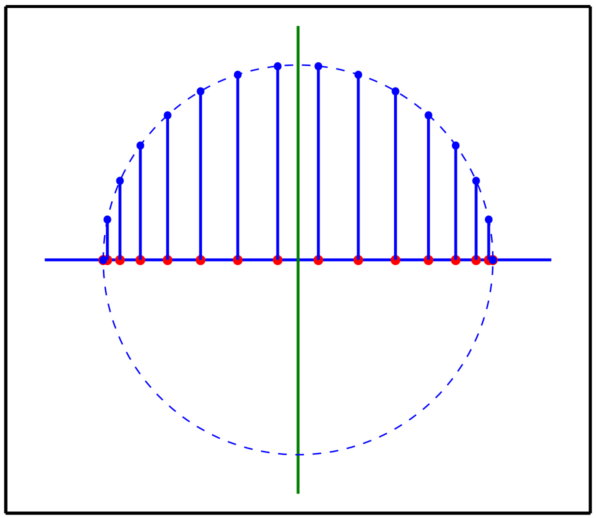
Chebyshev poly approx: 3 good reasons

- polynomial and Fourier approximation ("spectral approximation") converges faster than finite diff or finite elem or cubic splines or wavelets on analytic functions
- Though the coefficients in our DDE are periodic *the solutions are not*. Thus Fourier not so good.
  (Also: poly approx can be good on each piece of a piecewise-analytic fcn without generating Gibbs phenomena.)
- Chebyshev points are nearly optimal polynomial interpolation points for minimizing uniform error

Chebyshev collocation points

Chebyshev poly approx can be implemented by *collocation*.
Cheb collocation (i.e. spectral differentiation) can be implemented by FFT.
For degree N, Cheb collocation points are

$$t_j = \cos(j\pi/N), \qquad j = 0, \ldots, N.$$



Note $t_j \in [-1, 1]$. If needed, shift and scale the collocation points to interval $[0, \tau]$.

Spectral convergence of Cheb interpolation

Theorem. If f is analytic on $[-1,1]$ and if p_N is the Nth degree polynomial interpolant of f at the Cheb collocation points then there is $0 \leq \rho < 1$ and $C > 0$ so that

$$\|f - p_N\|_\infty \leq C\,\rho^N$$

for all $N \geq 1$.

Cheb collocation approx of U

$$\dot{y} = A(t)y + B(t)y_{-\tau} \text{ with } y(t) = \phi(t), t \in [-\tau, 0]$$

is approximated by

$$D_N v = M_A v + M_B w$$

where $v \approx y$, $w \approx \phi$ are numerical approximations,
$v, w \in \mathbb{C}^{N+1}$,
D_N is the spectral differentiation matrix, and
M_A, M_B are approximate multiplication operators.

Solving for v approximates U itself:

$$U \approx U_N \equiv (D_N - M_A)^{-1} M_B$$

Example: Eigs of a scalar DDE

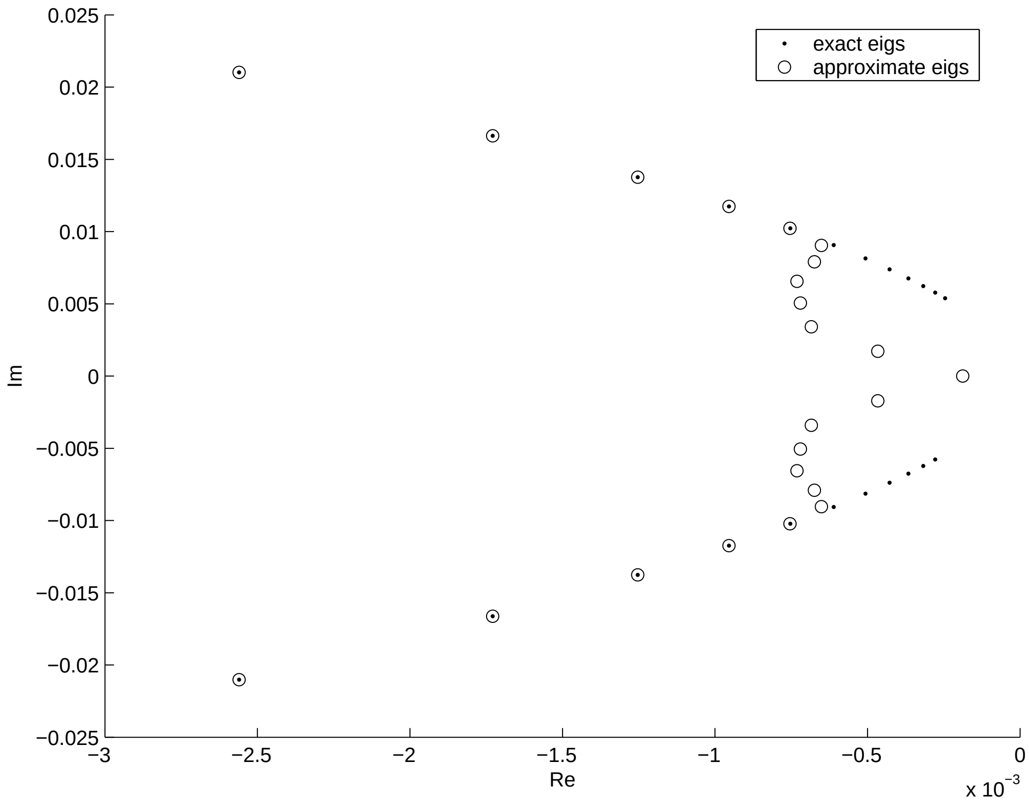
For $\dot{x} = -x + (1/2)x_{-1}$ we compare:

"Exact" method: Each root $\mu \in \mathbb{C}$ of characteristic eqn $\mu = -1 + 0.5e^{-\mu}$ is eigenvalue of U. Reduce char eqn to real variable problem. Solve by robust one-variable method (e.g. bisection) to 10^{-14} relative accuracy.

vs

Cheb collocation with $N = 29$: Compute U_N. Find eigs of U_N by QR iteration.

Example, cont



RESULT: Over 100 digits of correct eigenvalues from 30×30 matrix approx of U.
Only eigs near $0 \in \mathbb{C}$ are inaccurate (irrelevant for stability).

Cost of a stability chart

Using an $m \times m$ matrix approximation to U,

$$\textit{time to produce a chart} = O((\text{\# of pixels}) \cdot m^3)$$

with standard estimates on QR method for eigenvalues.

Thus m matters. For us, $m = d(N+1)$.

a posteriori estimate for DDE IVPs

Theorem. Consider IVP $\dot{y} = A(t)y + B(t)y_{-\tau}$, $y(t) = \phi(t)$ for $t \in [-\tau, 0]$. Find collocation solution p, a degree N polynomial. Then there is C which depend on $a(t)$ but is $O(1)$ in N so that

$$\begin{aligned}\|y - p\|_\infty \le C\Big(&\|I_N(Ap) - Ap\|_\infty \\ &+ \|I_N(B\phi) - B\phi\|_\infty \\ &+ |\dot{p}(0) - A(0)\phi(0) - B(0)\phi(-\tau)|\Big)\end{aligned}$$

where $I_N(f)$ is interpolating polynomial of f.

Thus error has two sources: (i) interpolation errors; (*ii*) residual error at initial time.

Right-hand side decays exponentially in N in practice.

a posteriori estimates of eigenvalues of U

Theorem. Let $N \geq 1$. Suppose coefficients $A(t), B(t)$ are analytic $d \times d$ matrix-valued functions.

Let $\delta > 0$. Suppose $Ux = \mu x$ for $x \in H^1$, $\|x\| = 1$, and $|\mu| \geq \delta$. Assume $U_N = V \Lambda V^{-1}$ with Λ diagonal. Let λ_i be the eigenvalues of U_N; order by decreasing magnitude.

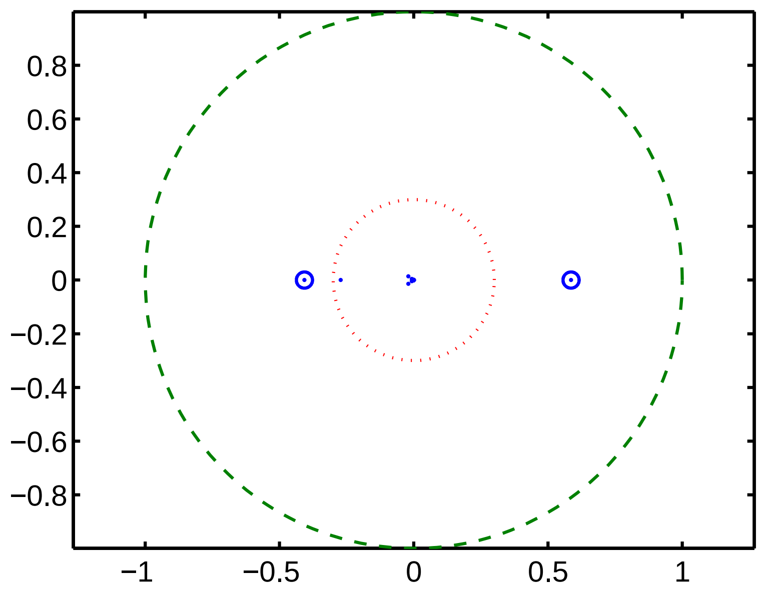
There are N positive quantities ω_k, which have *a posteriori* estimates, and which decay exponentially with N in practice, so that

$$\min |\mu - \lambda_i| \leq \min \{\omega_1, \ldots, \omega_N\} \operatorname{cond}(V).$$

Provable eigenvalues of U.

Example: Consider $\ddot{x} + \dot{x} + (1 + \cos(\pi t))x = 0.5x(t-2)$. This is a damped Mathieu equation. Let $N = 73$ and compute U_N. Apply previous theorem with $\delta = 0.3$.

Result: $\min |\mu - \lambda_i| \leq 0.03019$. In figure, dots are eigs of U_N; discs are *proven* error bounds for eigs bigger than δ.



Proposition (with *a posteriori* proof): If μ is an eig of U and $|\mu| \geq 0.3$ then μ is in one of the discs.

eigenvalue perturbation on Hilbert spaces

In our case $U \in \mathcal{L}(H^1)$ has finite rank approximation U_N, also acting on H^1. We diagonalize

$$U_N = V\Lambda V^{-1},$$

and we compute the condition number of V.

Theorem. (Bauer-Fike for Hilbert spaces) If $A, B \in \mathcal{L}(\mathcal{H})$, if $Bx = \mu x$ and $A = V\Lambda V^{-1}$ with Λ a multiplication operator on L^2 and V an isomorphism of Hilbert spaces then

$$\min_{\lambda\in\sigma(A)} |\mu - \lambda| \le \|V\| \|V\|^{-1} \|(B - A)x\|.$$

For us: $A = U_N$, $B = U$.

In our case Ux, $U_N x$ *are exact and approximate solns to IVPs*.

So $\|(U - U_N)x\|$ has an *a posteriori* estimate. In fact, ω_k in previous theorem are (proportional to) the values $\|(U - U_N)T_k\|$ where T_k is the kth Chebyshev polynomial.

That is, we prove that if one can solve the DDE accurately with the first N Chebyshev polynomials as initial functions then the numerical eigenvalues are accurate.

Why people really care about U

Interesting systems are *nonlinear* DDEs. (Linear, periodic DDEs are (usually) their linearizations about periodic orbits.)

Questions about nonlinear DDE:

1. were are periodic orbits?
2. nature of bifurcations?

Assuming question 1 is solved, for question 2 we need *good bases for stable and unstable directions*.

Good approximations to the monodromy operator of the linearization therefore desirable for nonlinear analysis, too.