Selected Solutions to Homeworks #1 and #2

Johnson 1.1. If $w \in C[0,1]$ and if $\int_0^1 wv\,dx = 0$ for all functions $v \in V = H_0^1$ then $w \equiv 0$ (i.e. $w(x) = 0$ for all $x \in [0,1]$).

Proof. By contradiction. We suppose w is not identically zero.

For instance, assume $A = w(a) > 0$ for some $a \in (0,1)$. Since w is continuous, there is $\epsilon > 0$ so that $w(x) \geq A/2$ for $x \in (a-\epsilon, a+\epsilon)$ and so that $(a-\epsilon, a+\epsilon) \subset [0,1]$. Let v be the continuous function which equals one in $(a-\epsilon/2, a+\epsilon/2)$, is zero outside $(a-\epsilon, a+\epsilon)$, and is linear between. Then

$$\int_0^1 wv\,dx \geq \int_{a-\epsilon/2}^{a+\epsilon/2} \frac{A}{2}\,dx = \frac{A}{2}\epsilon > 0,$$

a contradiction to $w(a) > 0$.

We have the essential proof already, but one must also consider the cases $a = 0$ and $a = 1$ as well as when $w(a) < 0$. Once those cases are covered we get a contradiction to w being nonzero somewhere. □

Johnson 1.3. Let $0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1$ be a partition of $I = [0,1]$. Let $I_j = [x_{j-1}, x_j]$ for $j = 1, \dots, M+1$. Describe a basis of "hat-like" functions for

$$V_h = \{v(x) : v \in C(I), v(0) = v(1) = 0, \text{ and } v \text{ is quadratic on each } I_j\}.$$

Formulate the finite element method for boundary value problem *(D)* on V_h, and fill in the details if the partition is uniform.

Solution. Consider the dimension of V_h. For each quadratic function on an interval I_j there are three unknown parameters. But the imposition of continuity, justified because we seek $V_h \subset H_0^1$, gives one constraint at each of the x_j, $j = 0, \dots, M+1$. Thus

$$\dim V_h = 3(M+1) - (M+2) = 2M+1.$$

Thus we need $2M+1$ "hat" functions.

Let $h_j = x_j - x_{j-1}$. I define

$$\psi_j(x) = \frac{4(x_j - x)(x - x_{j-1})}{h_j^2}, \quad j = 1, \dots, M+1$$

for $x \in I_j$ and zero otherwise. Define

$$\varphi_j(x) = \frac{(x_{j+1} - x)(x - x_{j-1})}{h_{j+1}h_j}, \quad j = 1, \dots, M$$

for $x \in I_{j-1} \cup I_j$ and zero otherwise. Note $\psi_j(x_k) = 0$ for *all* k and $\psi_j((x_{j-1} + x_j)/2) = 1$. On the other hand, $\varphi_j(x_k) = \delta_{jk}$. It is clear that $\{\psi_j, \varphi_j\}$ is a basis of V_h.

It is easy to write the form of the system $A\vec{\xi} = b$ corresponding to the FEM on *(D)* using this basis. Let $u_h = \sum_{j=1}^{M+1} \xi_j \psi_j + \sum_{j=M+2}^{2M+1} \xi_j \varphi_{j-M-1}$. Then the fundamental FEM principle

$$(u_h', v') = (f, v) \quad \text{for all} \quad v \in V_h,$$

which we impose by choosing v to be the basis elements, becomes

$$\sum_{j=1}^{M+1} \xi_j (\psi_j', \psi_k') + \sum_{j=M+2}^{2M+1} \xi_j (\varphi_{j-M-1}', \psi_k') = (f, \psi_k), \qquad k = 1, \dots, M+1,$$

$$\sum_{j=1}^{M+1} \xi_j (\psi_j', \varphi_k') + \sum_{j=M+2}^{2M+1} \xi_j (\varphi_{j-M-1}', \varphi_k') = (f, \varphi_k), \qquad k = 1, \dots, M.$$

The details of the above system are more calculable for a uniform partition with spacing h, but they still take a lot of paper. We might start by calculating that $\psi_j'(x) = \frac{1}{h^2}\left(-2x + (x_{j-1} + x_j)\right)$ and thus:

$$(\psi_j', \psi_k') = \delta_{jk} \int_{I_j} (\psi_j')^2 \, dx = \frac{\delta_{jk}}{h^4} \int_{x_{j-1}}^{x_j} \left(-2x + (x_{j+1} + x_j)\right)^2 \, dx = \dots$$

but my time is limited, too. □

Johnson 1.4. Formulate a finite difference method for *(D)* and compare to (1.6).

Solution. Let $0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1$ be a *uniform* partition of $I = [0,1]$ with $x_j - x_{j-1} = h = \frac{1}{M+1}$. Recall that $u''(x) = \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right) + O(h^2)$; see any numerical analysis book and note the $O(h^2)$ error depends on $u^{(iv)}$. Suppose w_j approximates $u(x_j)$ for $j = 1, \dots, M$. Then $-u'' = f$ is approximated by

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} = f(x_j)$$

for $j = 1, \dots, M$, as long as we interpret $w_0 = 0$ and $w_{M+1} = 0$, thus incorporating the boundary conditions. As a matrix equation this is

$$\text{(1)} \qquad \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_M \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_M) \end{pmatrix}.$$

Let's compare to (1.6). By multiplying (1) by h, the left sides are identical, but we note that in (1.6) the load elements are $b_j = (f, \varphi_j) = \int_{x_{j-1}}^{x_{j+1}} f(x)\varphi_j(x)\, dx$ while in the finite difference case we have $\tilde{b}_j = h f(x_j)$. It turns out that $b_j, \tilde{b}_j$ are both pretty good, but distinct, approximations to the integral

$$\int_{x_j - \frac{h}{2}}^{x_j + \frac{h}{2}} f(x)\, dx,$$

in the finite element case using a weighting (hat) function which extends outside of the interval $[x_j - \frac{h}{2}, x_j + \frac{h}{2}]$ and in the finite difference case by application of the midpoint rule. If $f(x)$ happens to be linear then the results will be identical; otherwise generally not. $\square$